Measuring the Price of Anarchy in Critical Care Units

Vince Knight, Cardiff University, @drvinceknight

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- $k = 1$
- $\mathcal{P}_1 = \{1, 2\}$
- $c_1 = 1$ and $c_2 = x$
- $r = 1$

The Nash flow minimises:

$$
\Phi(y, 1 - y) = \sum_{e=1}^{2} \int_{0}^{f_e} c_e(x)dx = \int_{0}^{y} 1dx + \int_{0}^{1-y} xdx
$$

$$
= y + \frac{(1 - y)^2}{2} = \frac{1}{2} + \frac{y^2}{2}
$$

$$
\Rightarrow \tilde{f} = (0, 1)
$$
Game Theory and Healthcare
Selsh Routing in Public Services in European Journal of Operational Research. 2013
Selfish Routing in Public Services in *European Journal of Operational Research*. 2013
What about the controllers?
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Mathematical modelling of patient flows to predict critical care capacity required following the merger of two District General Hospitals into one.
Mathematical modelling of patient flows to predict critical care capacity required following the merger of two District General Hospitals into one., *Anaesthesia*
\((K_{NH}, K_{RG}) = (6, 12):\)

\[
\begin{align*}
\text{\textbullet} & \quad \text{\textbullet} & \quad \text{\textbullet} \\
0.2 & \quad 0.4 & \quad 0.6 & \quad 0.8 & \quad 1
\end{align*}
\]

\(h = NH\)

\(h = RG\)

\((K_{NH}, K_{RG}) = (1, 12):\)

\[
\begin{align*}
\text{\textbullet} & \quad \text{\textbullet} & \quad \text{\textbullet} \\
0.2 & \quad 0.4 & \quad 0.6 & \quad 0.8 & \quad 1
\end{align*}
\]

\(h = NH\)

\(h = RG\)
For all $h \in \{NH, RG\}$ minimise:

$$(U_h - t)^2$$

Subject to:

$$0 \leq K_h \leq c_h$$

$$K_h \in \mathbb{Z}$$
\[ A = \begin{pmatrix} \left( U_{NH}(1, 1) - t \right)^2 & \ldots & \left( U_{NH}(1, c_{RG}) - t \right)^2 \\ \left( U_{NH}(2, 1) - t \right)^2 & \ldots & \left( U_{NH}(2, c_{RG}) - t \right)^2 \\ \vdots & \ddots & \vdots \\ \left( U_{NH}(c_{NH}, 1) - t \right)^2 & \ldots & \left( U_{NH}(c_{NH}, c_{RG}) - t \right)^2 \end{pmatrix} \]

\[ B = \begin{pmatrix} \left( U_{RG}(1, 1) - t \right)^2 & \ldots & \left( U_{RG}(1, c_{RG}) - t \right)^2 \\ \left( U_{RG}(2, 1) - t \right)^2 & \ldots & \left( U_{RG}(2, c_{RG}) - t \right)^2 \\ \vdots & \ddots & \vdots \\ \left( U_{RG}(c_{RG}, 1) - t \right)^2 & \ldots & \left( U_{RG}(c_{RG}, c_{RG}) - t \right)^2 \end{pmatrix} \]
Theorem.
Let \( f_h(k) : [1, c_h] \to [1, c_h] \) be the best response of player \( h \in \{\text{NH, RG}\} \) to the diversion threshold of \( \tilde{h} \neq h \) (\( \tilde{h} \in \{\text{NH, RG}\} \)). If \( f_h(k) \) is a non-decreasing function in \( k \) then the game has at least one Nash Equilibrium in Pure Strategies.
Lemma.

- If $\lambda^{(a)}_{NH} \leq \lambda^{(b)}_{NH}$ and $\lambda^{(c)}_{NH} \leq \lambda^{(d)}_{NH}$ then $f_{NH}(k)$ is a non-decreasing function in $k$.

- If $\lambda^{(a)}_{RG} \leq \lambda^{(c)}_{RG}$ and $\lambda^{(b)}_{RG} \leq \lambda^{(d)}_{RG}$ then $f_{RG}(k)$ is a non-decreasing function in $k$. 
$U_{NH}$

- $f_{NH}(3) = m$
- $f_{NH}(4) = m + 1$
- $f_{NH}(5) = m + 1$

Legend:
- $\times\times K_{NH} = m + 1$
- $\blacklozenge K_{NH} = m$
- $\bullet K_{NH} = m - 1$
- $\blacksquare K_{NH} = m + 2$
Best response to RG
Best response to NH

(t = 0.8)  
(t = 0.6)
PoA = \frac{T^*}{\tilde{T}}
$\arg\min_t (\text{PoA}(x))$
Conclusions
- Developed a strategic form game representation of CCU interaction;
- Proved structural properties of equilibrium behaviour;
- Identified a potential justified approach to obtaining policies.
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Proved structural properties of equilibrium behaviour;

Identified a potential justified approach to obtaining policies.

Measuring the Price of Anarchy in Critical Care Unit Interactions, *Minor revisions to Journal of Operational Research*
@drvinceknight
knightva@cardiff.ac.uk
vknighth.org/Talks

@IzabelaKomenda
Professor Jeff Griffiths