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Academic Year: 2025/26

Examination Period: Autumn

Module Code: MA3604

Examination Paper Title: Game Theory

Duration: 2 hours

Please read the following information carefully:

Structure of Examination Paper:

- There are 5 pages including this page.
- There are **4** questions in total.
- There are no appendices.
- The maximum mark for the examination paper is 75 and the mark obtainable for a question or part of a question is shown in brackets alongside the question.

Instructions for completing the examination:

- Complete the front cover of any answer books used.
- This examination paper must be submitted to an Invigilator at the end of the examination.
- Answer **THREE** questions.
- Each question should be answered on a separate page.

You will be provided with / or allowed:

- **ONE** answer book.
- The **use of calculators** is **permitted** in this examination.
- The use of a translation dictionary between English or Welsh and another language, provided that it bears an appropriate departmental stamp, is permitted in this examination.

1. (a) Provide definitions for the following terms:

- Normal form game.

Solution:
Book Work. [1]

- Strictly dominated strategy.

Solution:
Book Work. [1]

- Weakly dominated strategy.

Solution:
Book Work. [1]

- Best response strategy.

Solution:
Book Work. [1]

- Nash equilibrium.

Solution:
Book Work. [1]

[5]

(b) Consider the following Normal Form Game defined by:

$$M_r = \begin{pmatrix} 7 & 0 \\ 2 & 6 \end{pmatrix} \qquad M_c = \begin{pmatrix} 3 & 2 \\ 0 & 2 \end{pmatrix}$$

Where M_r corresponds to the payoffs of the row player and M_c corresponds to the payoffs of the column player.

(i) Obtain all (if any) pure Nash equilibria (where each player chooses a single action with certainty). [5]

Solution:
Identifying the pure action best responses:

$$M_r = \begin{pmatrix} \underline{7} & 0 \\ 2 & \underline{6} \end{pmatrix} \qquad M_c = \begin{pmatrix} \underline{3} & 2 \\ 0 & \underline{2} \end{pmatrix}$$

[4]

This gives two pure NE:

$$\{((1,0),(1,0)),((0,1),(0,1))\}$$

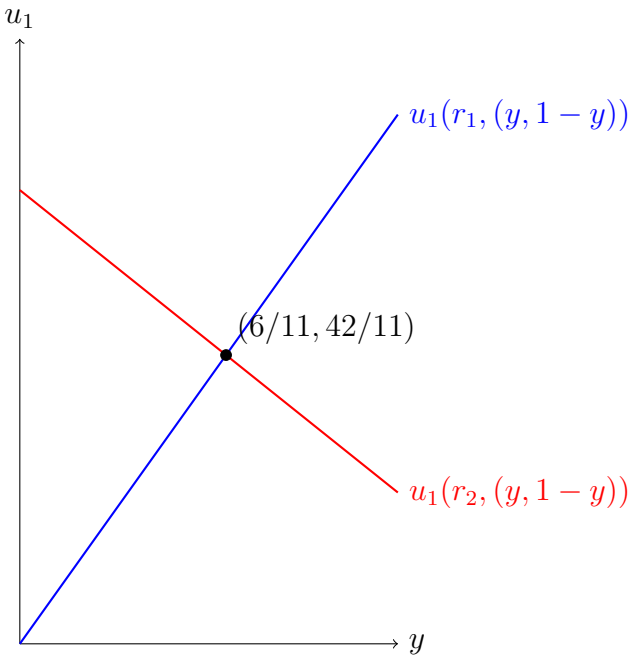
[1]

(ii) Sketch the expected utilities to player 1 (the row player) of each action, assuming that the 2nd player (the column player) plays a strategy: $\sigma_2 = (y, 1-y)$. [4]

Solution:
We have:

$$u_1(r_1, (y, 1-y)) = 7y \qquad u_1(r_2, (y, 1-y)) = 2y + 6 - 6y = 6 - 4y$$

[2]



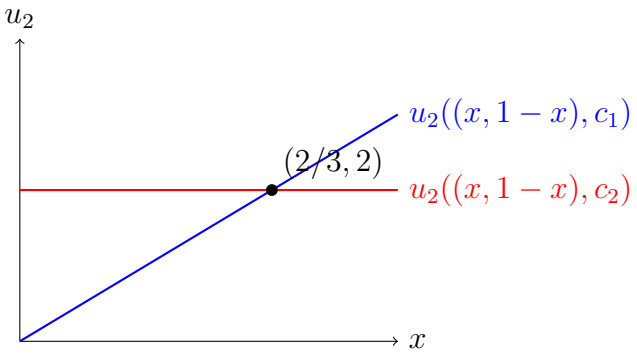
[2]

(iii) Sketch the expected utilities to player 2 (the column player) of each action, assuming that the 1st player (the row player) plays a strategy: $\sigma_1 = (x, 1-x)$. [4]

Solution:
We have:

$$u_2((x, 1-x), c_1) = 3x \qquad u_2((x, 1-x), c_2) = 2$$

[2]



[2]

(iv) State and prove the best response condition theorem. Using this or otherwise obtain all Nash equilibria for the game. Confirm this using your answers to question (ii) and (iii).

[7]

Solution:

State and prove theorem is bookwork. [3]

Using the theorem corresponds to the support enumeration algorithm. We have found all NE with support size 1 (the pure NE). Now for support size 2.

For σ_2:

$$7y = 6 - 4y$$

gives:

$$y = 6/11$$

so $\sigma_1 = (6/11, 5/11)$, as there are only two actions there is no further support to explore. [1]

For σ_1:

$$3x = 2$$

gives:

$$x = 2/3$$

so $\sigma_2 = (2/3, 1/3)$ [1]

From the plots from previous questions we have:

$$\sigma_1^* \begin{cases} (0,1) & y < 6/11 \\ (1,0) & y > 6/11 \\ \text{indifferent} & y = 6/11 \end{cases} \qquad \sigma_2^* \begin{cases} (0,1) & x < 2/3 \\ (1,0) & x > 1/3 \\ \text{indifferent} & x = 2/3 \end{cases}$$

[1]

We see 3 pairs of best responses:

$$\{((1,0),(1,0)),((0,1),(0,1)),((2/3,1/3),(6/11,5/11))\}$$

[1]

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2. Throughout, the following game is considered:

Road users in a given country can choose to drive on either the left (L) side or the right (R) side of the road. The action set in this game is $\mathcal{A} = \{L, R\}$.

If all users drive on the same side of the road then no accidents will occur. If users drive on the opposite side and meet each other then they may have an accident.

Considering a population vector $x = (x_1, 1 - x_1)$ where x_1 is the proportion of the population using action L, the fitness functions are given by:

$$f_L(x) = 1 + x_1$$

and

$$f_R(x) = 1 + (1 - x_1)$$

(a) Give an interpretation for the fitness functions. [1]

Solution:

$$f_1(x) = \underbrace{1}_{\text{Base fitness of not crashing}} + \underbrace{x_1}_{\text{Added utility representing the decreased chance of crashing.}}$$

$$f_2(x) = \underbrace{1}_{\text{Base fitness of not crashing}} + \underbrace{(1 - x_1)}_{\text{Added utility representing the decreased chance of crashing.}}$$

[1]

(b) State the replicator dynamics equation.

Solution:
Bookwork. [1]

[1]

(c) Define a stable population.

Solution:
Bookwork. [1]

[1]

(d) Define a post entry population.

Solution:
Bookwork. [2]

[2]

(e) Define an evolutionary stable strategy.

Solution:
Bookwork. [2]

[2]

(f) Give the replicator dynamics equation for this game and obtain all stable populations. [10]

Solution:
The replicator dynamics equation gives:

$$\frac{dx_1}{dt} = x_1(f_1(x_1) - \phi)$$

where

$$\begin{aligned} \phi &= x_1 f_1(x) + (1 - x_1) f_2(x) && (1)\\ &= x_1(1 + x_1) + (1 - x_1)(1 + 1 - x_1) && (2)\\ & && (3) \end{aligned}$$

[4]

so the replicator dynamics equation is:

$$\begin{aligned} \frac{dx_1}{dt} &= x_1(1 + x_1 - x_1(1 + x_1) - (1 - x_1)(2 - x_1)) && (4)\\ &= x_1(1 - x_1^2 - 2(1 - x_1) + (1 - x_1)x_1) && (5)\\ &= x_1(-2x_1^2 + 3x_1 - 1) && (6)\\ &= x_1(1 - x_1)(2x_1 - 1) && (7)\\ & && (8) \end{aligned}$$

[4]

There are thus 3 stable populations:

$$\{(0, 1), (1, 0), (1/2, 1/2)\}$$

[2]

(g) Using an initial population of $x = (0.49, 0.51)$, calculate the first 2 steps of Euler's method of numerical integration with a step size of $h = 0.01$.

[7]

Solution:

$$x_1^{(0)} = .49$$

First step:

$$\begin{aligned} x_1^{(1)} &= x_1^{(0)} + h \cdot \frac{dx_1}{dt}|_{x_1=.49} && (9)\\ &= .49 + .01 \cdot .49 \cdot .51 \cdot (2 \cdot .49 - 1) && (10)\\ &\approx .48995 && (11) \end{aligned}$$

[4]

Second step:

$$\begin{aligned} x_1^{(2)} &= x_1^{(1)} + h \cdot \frac{dx_1}{dt}|_{x_1=.48995} && (12)\\ &= .48995 + .01 \cdot .48995 \cdot .51005 \cdot (2 \cdot .48995 - 1) && (13)\\ &\approx 0.48989977 && (14) \end{aligned}$$

[3]

(h) Offer an interpretation for the answer to question (g) and interpret this in terms of evolutionary stable strategies. [1]

Solution:

This shows that a small deviation from $(1/2, 1/2)$ does not return to $(1/2, 1/2)$ as $x_1^{(2)} < x_1^{(1)} < x_1^{(0)}$. This indicates that it is not an Evolutionary Stable Strategy although further investigation would be needed to confirm that a smaller deviation does not give a different outcome. [1]

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3. (a) Define a characteristic function game $G = (N, v)$.

[2]

Solution:
Bookwork. [2]

(b) Define the Shapley value.

[2]

Solution:
Bookwork. [2]

(c) Obtain the Shapley value for the following characteristic function games:

$$v_1(c) = \begin{cases} 0, & \text{if } c = \emptyset \\ 6, & \text{if } c = \{1\} \\ 6, & \text{if } c = \{2\} \\ 7, & \text{if } c = \{3\} \\ 7, & \text{if } c = \{1,2\} \\ 7, & \text{if } c = \{2,3\} \\ 20, & \text{if } c = \{1,3\} \\ 40, & \text{if } c = \{1,2,3\} \end{cases} \qquad v_2(c) = \begin{cases} 0, & \text{if } c = \emptyset \\ 100, & \text{if } c = \{1\} \\ 6, & \text{if } c = \{2\} \\ 7, & \text{if } c = \{3\} \\ 100, & \text{if } c = \{1,2\} \\ 7, & \text{if } c = \{2,3\} \\ 100, & \text{if } c = \{1,3\} \\ 100, & \text{if } c = \{1,2,3\} \end{cases}$$

[8]

Solution:
For v_1, we have:

$$\Delta^G_{(1,2,3)} = (6, 1, 33) \tag{15}$$
$$\Delta^G_{(1,3,2)} = (6, 20, 14) \tag{16}$$
$$\Delta^G_{(2,1,3)} = (1, 6, 33) \tag{17}$$
$$\Delta^G_{(2,3,1)} = (33, 6, 1) \tag{18}$$
$$\Delta^G_{(3,1,2)} = (13, 20, 7) \tag{19}$$
$$\Delta^G_{(3,2,1)} = (33, 0, 7) \tag{20}$$
$$\tag{21}$$

[3]

Which gives:

$$\phi(G) = (46/6, 53/6, 95/6)$$

[1]

For v_2, we have:

$$\Delta^G_{(1,2,3)} = (100, 0, 0) \tag{22}$$
$$\Delta^G_{(1,3,2)} = (100, 0, 0) \tag{23}$$
$$\Delta^G_{(2,1,3)} = (94, 6, 0) \tag{24}$$
$$\Delta^G_{(2,3,1)} = (93, 6, 1) \tag{25}$$
$$\Delta^G_{(3,1,2)} = (93, 0, 7) \tag{26}$$
$$\Delta^G_{(3,2,1)} = (93, 0, 7) \tag{27}$$
$$\tag{28}$$

[3]

Which gives:

$$\phi(G) = (191/2, 2, 5/2)$$

[1]

(d) For a given characteristic function game $G = (N, v)$ a payoff vector λ is efficient if:

$$\sum_{i=1}^{N} \lambda_i = v(\Omega)$$

Prove that the Shapley value is efficient.

[6]

Solution:
If $\lambda_i = \phi_i(G)$:

$$\sum_{i=1}^{N} \lambda_i = \sum_{i=1}^{N} \frac{1}{N!} \sum_{\pi \in \Pi_n} \Delta^G_\pi(i) \tag{29}$$
$$= \frac{1}{N!} \sum_{\pi \in \Pi_n} \sum_{i=1}^{N} \Delta^G_\pi(i) \tag{30}$$
$$= \frac{1}{N!} \sum_{\pi \in \Pi_n} v(S_\pi(1)) - v(S_\pi(\emptyset)) + v(S_\pi(1,2)) - v(S_\pi(1)) + \dots \tag{31}$$
$$+ v(S_\pi(1, 2, \dots N)) - v(S_\pi(1, 2, \dots N-1)) \tag{32}$$
$$= \frac{1}{N!} \sum_{\pi \in \Pi_n} v(\Omega) = v(\Omega) \tag{33}$$

[6]

(e) The additivity property is:
Given two games $G_1 = (N, v_1)$ and $G_2 = (N, v_2)$, define their sum $G^+ = (N, v^+)$ by:

$$v^+(C) = v_1(C) + v_2(C) \quad \text{for all } C \subseteq \Omega$$

A payoff vector λ satisfies the additivity property if:

$$\lambda_i^{(G^+)} = \lambda_i^{(G_1)} + \lambda_i^{(G_2)} \text{ for all } 1 \leq i \leq N$$

Using the two games from part (c) demonstrate that the Shapley value has the additivity property.

[7]

Solution:
Let us first calculate v^+:

$$v^+(c) = \begin{cases} 0, & \text{if } c = \emptyset \\ 106, & \text{if } c = \{1\} \\ 12, & \text{if } c = \{2\} \\ 14, & \text{if } c = \{3\} \\ 107, & \text{if } c = \{1,2\} \\ 14, & \text{if } c = \{2,3\} \\ 120, & \text{if } c = \{1,3\} \\ 140, & \text{if } c = \{1,2,3\} \end{cases}$$

[2] We have:

$$\Delta^G_{(1,2,3)} = (106, 1, 33) \tag{34}$$
$$\Delta^G_{(1,3,2)} = (106, 20, 14) \tag{35}$$
$$\Delta^G_{(2,1,3)} = (95, 12, 33) \tag{36}$$
$$\Delta^G_{(2,3,1)} = (126, 12, 2) \tag{37}$$
$$\Delta^G_{(3,1,2)} = (106, 20, 14) \tag{38}$$
$$\Delta^G_{(3,2,1)} = (126, 0, 14) \tag{39}$$
$$\tag{40}$$

[3]

Which gives:

$$\phi(G) = (665/6, 65/6, 55/3)$$

[1]

From the previous questions (for v^1 and v^2) if the additivity property holds we have:

$$\phi(G) = (46/6, 53/6, 95/6) + (191/2, 2, 5/2) = (665/6, 65/6, 55/3)$$

as required. [1]

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4. Consider the following matching game involving three suitors S_1, S_2, S_3 and three reviewers R_1, R_2, R_3. Each person ranks all members of the opposite group in strict order of preference:

$$f(S1) = (R2, R1, R3)$$
$$f(S2) = (R1, R3, R2)$$
$$f(S3) = (R1, R3, R2)$$
$$g(R1) = (S1, S2, S3)$$
$$g(R2) = (S2, S1, S3)$$
$$g(R3) = (S2, S3, S1)$$

(a) Use the Gale-Shapley algorithm to compute the suitor-optimal stable matching. Clearly show each step of the algorithm. [9]

Solution:

Let us apply the Gale-Shapley algorithm to this set of preferences.

We pick $S1$ and assign the top of their preference list:

$$M(S1) = R2$$

[2]

We now pick $S2$ and assign the top of their preference list:

$$M(S1) = R2 \qquad M(S2) = R1$$

[2]

We now pick $S3$, the top of their preference list is $R1$ which is matched. $R1$ prefers their current matching over $S3$. We remove $R1$ from $S3$'s preference list:

$$f(S3) \to (R3, R2)$$

[2]

We pick $S3$ again (the only unmatched suitor), the top of their preference list is $R3$ which is unmatched giving: [2]

$$M(S1) = R2 \qquad M(S2) = R1 \qquad M(S3) = R3$$

[1]

(b) Consider the following situation: after the matching is publicly announced and assumed to be binding, suitor $S3$ considers making a private offer to reviewer $R1$ to persuade them to leave their assigned match and instead pair with them.

This interaction is modeled as the following extensive form game:

- Stage 1: $S3$ chooses whether to make the offer at a cost of r with $r \neq 1$ and $r < 2$, which is paid regardless of whether or not the offer is accepted.
- Stage 2: If the offer is made, $R1$ chooses whether to accept.
- If $R1$ accepts, they pair with $S3$. If they decline, the original matching holds.

Each person's utility is defined as follows:

- A player receives 2 if matched with their first choice, 1 if matched with their second choice, and 0 if matched with their third choice or unmatched.
- $S3$ incurs a cost of r if they makes the offer (regardless of the outcome).
- $R1$ obtains a benefit r if they accept the offer.

Draw the extensive form game (game tree), assuming the initial matching is the stable matching you found in part (a).

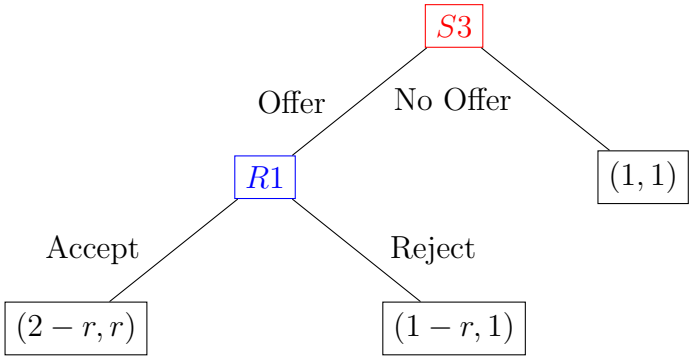
[7]

Solution:

The utilities of each matching is:

- $S3$: $R1 \to 2$, $R3 \to 1$, $R2 \to 0$
- $R1$: $S1 \to 2$, $S2 \to 1$, $S3 \to 0$

[1]



[6]

(c) Find the subgame perfect equilibrium of this game. Justify your answer.

[5]

Solution:

To use backwards induction, which is known to give a subgame perfect equilibrium two cases must be considered: [1]

$r > 1$:

The reviewer will choose Accept (as $r > 1$) however the suitor will not make the Offer as $2 - r < 1$.

(No Offer, Accept) [2]

$r < 1$:

The suitor will Reject (as $r < 1$) and the reviewer will make the Not Offer as $r < 1$. (No Offer, Reject) [2]

(d) Briefly interpret the result: does the potential for a side offer undermine the stability of the original matching? [4]

Solution:

In this particular case, the stability of the matching is not undermined. [1] This seems to go in line with the theorem guaranteeing the stability of the output of the Gale-Shapley Algorithm. [2]. However, given a slightly different preference function for either the reviewer or the suitor this might not be the case. [1]
