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Academic Year: 2025/26

Examination Period: Autumn

Module Code: MA3604

Examination Paper Title: Game Theory

Duration: 2 hours

Please read the following information carefully:

Structure of Examination Paper:

- There are 5 pages including this page.
- There are **4** questions in total.
- There are no appendices.
- The maximum mark for the examination paper is 75 and the mark obtainable for a question or part of a question is shown in brackets alongside the question.

Instructions for completing the examination:

- Complete the front cover of any answer books used.
- This examination paper must be submitted to an Invigilator at the end of the examination.
- Answer **THREE** questions.
- Each question should be answered on a separate page.

You will be provided with / or allowed:

- **ONE** answer book.
- The **use of calculators** is **permitted** in this examination.
- The use of a translation dictionary between English or Welsh and another language, provided that it bears an appropriate departmental stamp, is permitted in this examination.

1. (a) Provide definitions for the following terms:

- Normal form game.
- Strictly dominated strategy.
- Weakly dominated strategy.
- Best response strategy.
- Nash equilibrium.

[5]

(b) Consider the following Normal Form Game defined by:

$$M_r = \begin{pmatrix} 7 & 0 \\ 2 & 6 \end{pmatrix} \qquad M_c = \begin{pmatrix} 3 & 2 \\ 0 & 2 \end{pmatrix}$$

Where M_r corresponds to the payoffs of the row player and M_c corresponds to the payoffs of the column player.

(i) Obtain all (if any) pure Nash equilibria (where each player chooses a single action with certainty). [5]

(ii) Sketch the expected utilities to player 1 (the row player) of each action, assuming that the 2nd player (the column player) plays a strategy: $\sigma_2 = (y, 1-y)$. [4]

(iii) Sketch the expected utilities to player 2 (the column player) of each action, assuming that the 1st player (the row player) plays a strategy: $\sigma_1 = (x, 1-x)$. [4]

(iv) State and prove the best response condition theorem. Using this or otherwise obtain all Nash equilibria for the game. Confirm this using your answers to question (ii) and (iii).

[7]

2. Throughout, the following game is considered:

Road users in a given country can choose to drive on either the left (L) side or the right (R) side of the road. The action set in this game is $\mathcal{A} = \{L, R\}$.

If all users drive on the same side of the road then no accidents will occur. If users drive on the opposite side and meet each other then they may have an accident.

Considering a population vector $x = (x_1, 1 - x_1)$ where x_1 is the proportion of the population using action L, the fitness functions are given by:

$$f_L(x) = 1 + x_1$$

and

$$f_R(x) = 1 + (1 - x_1)$$

(a) Give an interpretation for the fitness functions. [1]

(b) State the replicator dynamics equation. [1]

(c) Define a stable population. [1]

(d) Define a post entry population. [2]

(e) Define an evolutionary stable strategy. [2]

(f) Give the replicator dynamics equation for this game and obtain all stable populations. [10]

(g) Using an initial population of $x = (0.49, 0.51)$, calculate the first 2 steps of Euler's method of numerical integration with a step size of $h = 0.01$. [7]

(h) Offer an interpretation for the answer to question (g) and interpret this in terms of evolutionary stable strategies. [1]

3. (a) Define a characteristic function game $G = (N, v)$.

[2]

(b) Define the Shapley value.

[2]

(c) Obtain the Shapley value for the following characteristic function games:

$$v_1(c) = \begin{cases} 0, & \text{if } c = \emptyset \\ 6, & \text{if } c = \{1\} \\ 6, & \text{if } c = \{2\} \\ 7, & \text{if } c = \{3\} \\ 7, & \text{if } c = \{1,2\} \\ 7, & \text{if } c = \{2,3\} \\ 20, & \text{if } c = \{1,3\} \\ 40, & \text{if } c = \{1,2,3\} \end{cases} \qquad v_2(c) = \begin{cases} 0, & \text{if } c = \emptyset \\ 100, & \text{if } c = \{1\} \\ 6, & \text{if } c = \{2\} \\ 7, & \text{if } c = \{3\} \\ 100, & \text{if } c = \{1,2\} \\ 7, & \text{if } c = \{2,3\} \\ 100, & \text{if } c = \{1,3\} \\ 100, & \text{if } c = \{1,2,3\} \end{cases}$$

[8]

(d) For a given characteristic function game $G = (N, v)$ a payoff vector λ is efficient if:

$$\sum_{i=1}^{N} \lambda_i = v(\Omega)$$

Prove that the Shapley value is efficient.

[6]

(e) The additivity property is:
Given two games $G_1 = (N, v_1)$ and $G_2 = (N, v_2)$, define their sum $G^+ = (N, v^+)$ by:

$$v^+(C) = v_1(C) + v_2(C) \quad \text{for all } C \subseteq \Omega$$

A payoff vector λ satisfies the additivity property if:

$$\lambda_i^{(G^+)} = \lambda_i^{(G_1)} + \lambda_i^{(G_2)} \text{ for all } 1 \leq i \leq N$$

Using the two games from part (c) demonstrate that the Shapley value has the additivity property.

[7]

4. Consider the following matching game involving three suitors S_1, S_2, S_3 and three reviewers R_1, R_2, R_3. Each person ranks all members of the opposite group in strict order of preference:

$$
\begin{aligned}
f(S1) &= (R2, R1, R3) \\
f(S2) &= (R1, R3, R2) \\
f(S3) &= (R1, R3, R2) \\
g(R1) &= (S1, S2, S3) \\
g(R2) &= (S2, S1, S3) \\
g(R3) &= (S2, S3, S1)
\end{aligned}
$$

(a) Use the Gale-Shapley algorithm to compute the suitor-optimal stable matching. Clearly show each step of the algorithm. [9]

(b) Consider the following situation: after the matching is publicly announced and assumed to be binding, suitor $S3$ considers making a private offer to reviewer $R1$ to persuade them to leave their assigned match and instead pair with them.

This interaction is modeled as the following extensive form game:

- Stage 1: $S3$ chooses whether to make the offer at a cost of r with $r \neq 1$ and $r < 2$, which is paid regardless of whether or not the offer is accepted.
- Stage 2: If the offer is made, $R1$ chooses whether to accept.
- If $R1$ accepts, they pair with $S3$. If they decline, the original matching holds.

Each person's utility is defined as follows:

- A player receives 2 if matched with their first choice, 1 if matched with their second choice, and 0 if matched with their third choice or unmatched.
- $S3$ incurs a cost of r if they makes the offer (regardless of the outcome).
- $R1$ obtains a benefit r if they accept the offer.

Draw the extensive form game (game tree), assuming the initial matching is the stable matching you found in part (a).

[7]

(c) Find the subgame perfect equilibrium of this game. Justify your answer.

[5]

(d) Briefly interpret the result: does the potential for a side offer undermine the stability of the original matching? [4]