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**Academic Year:** MOCK

**Examination Period:** Autumn

**Module Code:** MA3604

**Examination Paper Title:** Game Theory

**Duration:** 2 hours

**Please read the following information carefully:**

**Structure of Examination Paper:**

- There are 7 pages including this page.
- There are **4** questions in total.
- There are no appendices.
- The maximum mark for the examination paper is 75 and the mark obtainable for a question or part of a question is shown in brackets alongside the question.

**Instructions for completing the examination:**

- Complete the front cover of any answer books used.
- This examination paper must be submitted to an Invigilator at the end of the examination.
- Answer **THREE** questions.
- Each question should be answered on a separate page.

**You will be provided with / or allowed:**

- **ONE** answer book.
- The **use of calculators** is **permitted** in this examination.
- The use of a translation dictionary between English or Welsh and another language, provided that it bears an appropriate departmental stamp, is permitted in this examination.

**1.** (a) Provide definitions for the following terms:

- Normal form game.
- Strictly dominated strategy.
- Weakly dominated strategy.
- Best response strategy.
- Nash equilibrium.

[5]

(b) Consider the following Normal Form Game defined by:

$$M_r = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \qquad M_c = \begin{pmatrix} 3 & 4 \\ 3 & 2 \end{pmatrix}$$

Where $M_r$ corresponds to the payoffs of the row player and $M_c$ corresponds to the payoffs of the column player.

(i) Obtain all (if any) pure Nash equilibria (where each player chooses a single action with certainty). [5]

(ii) Sketch the expected utilities to player 1 (the row player) of each action, assuming that the 2nd player (the column player) plays a strategy: $\sigma_2 = (y, 1-y)$. [4]

(iii) Sketch the expected utilities to player 2 (the column player) of each action, assuming that the 1st player (the row player) plays a strategy: $\sigma_1 = (x, 1-x)$. [4]

(iv) State and prove the best response condition theorem. Using this or otherwise obtain all Nash equilibria for the game. Confirm this using your answers to question (ii) and (iii).

[7]

**2.** At a local **indie game convention**, two groups interact: **developers** and **reviewers**.

- Each **developer** chooses whether to **Polish** their demo ($P$) or **Rush** to release it quickly ($R$).
- Each **reviewer** decides whether to **Highlight Indies** ($H$) or **Focus on Big Titles** ($B$).

If a developer polishes their demo and a reviewer highlights indies, both gain: the developer earns attention, and the reviewer builds a reputation for discovering quality games. If the reviewer focuses on big titles, a polished indie receives less benefit, while rushed games may still get lucky exposure. Reviewers, however, gain more visibility by covering popular titles — even if the games themselves are weaker.

The interaction can be represented by two payoff matrices, one for each group.

$$M_r = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}, \qquad M_c = \begin{pmatrix} 3 & 4 \\ 5 & 1 \end{pmatrix},$$

where $M_r$ gives the payoffs to the **developers** (rows) and $M_c$ gives the payoffs to the **reviewers** (columns).

Let:

- $x_P$ and $x_R = 1 - x_P$ denote the proportions of polishing and rushing developers;
- $y_H$ and $y_B = 1 - y_H$ denote the proportions of reviewers who highlight indies or focus on big titles.

(a) Compute the expected payoff of each strategy for both developers and reviewers. [2]

(b) Hence show that this interaction is modelled by the **asymmetric replicator dynamics:**

$$\frac{dx_P}{dt} = -2x_P^2 y_H + x_P^2 + 2x_P y_H - x_P$$
$$\frac{dy_H}{dt} = 5x_P y_H^2 - 5x_P y_H - 4y_H^2 + 4y_H$$

[7]

(c) Confirm that $\{\sigma_r = (4/5, 1/5), \sigma_c = (1/2, 1/2)\}$ is a Nash equilibrium for the game defined by the payoff matrices. [4]

(d) Confirm that this Nash equilibrium corresponds to a stable population.

(e) For the asymmetric replicator dynamics equation for a pair of population vectors $(x, y)$, some $\epsilon_x > 0$, $\epsilon_y > 0$ and a pair of strategies $\tilde{x}, \tilde{y})$ a post entry population pair is given by:

$$x_\epsilon = x + \epsilon_x(\tilde{x} - x) \qquad y_\epsilon = y + \epsilon_y(\tilde{y} - y)$$



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[2]

Using this, propose a definition for an evolutionary stable strategy pair for the asymmetric replicator dynamics equation. [6]

(f) Discuss how you would use an appropriate numerical integration technique to explore the stability of points near the initial population. **You are not expected to carry out any calculations.** [4]

**3.** Consider the stage game defined by:

$$M_r = \begin{pmatrix} 2 & 5 \\ 0 & 4 \end{pmatrix} \qquad M_c = \begin{pmatrix} 2 & 0 \\ 5 & 4 \end{pmatrix}$$

The first action for both players will be referred to as 'Cooperate' ($C$) and the second action will be referred to as 'Defect' ($D$). **Players aim to minimise their payoffs.**

Consider the following strategies for the infinitely repeated game with the above stage game.

- $s_C$: Always cooperate;
- $s_D$: Always defect;
- $s_G$: Start by cooperating until your opponent defects at which point defect in all future stages.

Assume $A_1 = A_2 = \{s_C, s_D, s_G\}$.

(a) Assuming a discounting factor of $\delta$, obtain the utility to both players if the action pair $(s_C, s_C)$ is played.

[2]

(b) Assuming a discounting factor of $\delta$, obtain the utility to both players if the action pair $(s_D, s_D)$ is played.

[2]

(c) For what values of $\delta$ is $(s_G, s_G)$ a Nash equilibrium? **Recall that players aim to minimise their payoffs.**

[5]

(d) Define the average payoff in an infinitely repeated game.

[1]

(e) Plot the feasible average payoffs and the individually rational payoffs for the game. **Recall that players aim to minimise their payoffs.**

[4]

(f) Prove the following theorem (**for games where players aim to minimise their payoffs**):

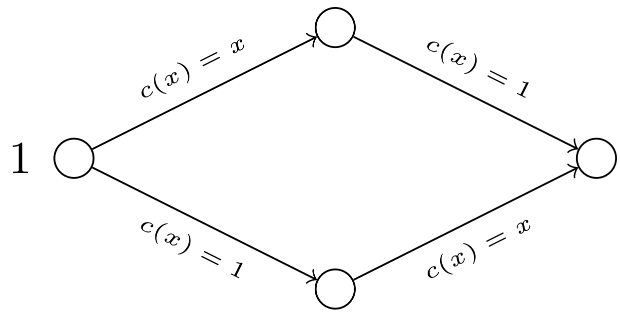
"Let $u_1^*, u_2^*$ be a pair of Nash equilibrium payoffs for a stage game. For every individually rational pair $v_1, v_2$ there exists $\bar{\delta}$ such that for all $1 > \delta > \bar{\delta} > 0$ there is a subgame perfect Nash equilibrium with payoffs $v_1, v_2$."

[11]

**4.** (a) Define a routing game $(G, r, c)$.

[2]

(b) Define a Nash flow and using this definition obtain the Nash flow for the following game:




[3]

(c) Define an optimal flow and using this definition obtain the optimal flow for the above game.

[3]

(d) State the theorem connecting the following function $\Phi$ to the Nash flow of a routing game:

$$\Phi(f) = \sum_{e\in E} \int_0^{f_e} c_e(x)dx$$

[2]

(e) Using the theorem from (d) confirm the Nash flow previously found in (b).

[2]

(f) State the theorem connecting the marginal cost $c^*(x) = \frac{d(xc(x))}{dx}$ to the optimal flow of a routing game.

[2]

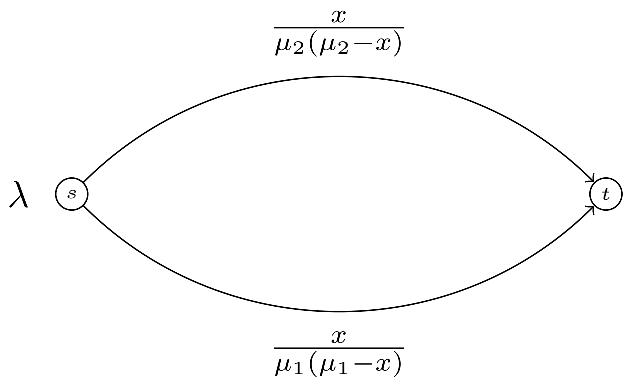
(g) Using the theorem from (f) confirm the optimal flow previously found in (c) .

[2]

(h) The expected time spent in an $M/M/1$ queue at steady state is given by:

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)}$$

where $\mu, \lambda$ are the mean service and inter arrival rates and $\lambda < \mu$ respectively. Explain how a system with two $M/M/1$ queues and players choosing which queue to join can be studied using the following routing game:




[2]

(i) Obtain the Nash and Optimal flows for the game in (h) with $\mu_1 = 4, \mu_2 = 3$ and $\lambda = 2$.

You might find it useful to know that the equation:

$$x^4 - 2x^3 + x^2 - 420x + 324 = 0$$

has a single solution in the range $0 \leq x < 2$ given by $x \approx 0.7715$.

[7]