1. (a) Provide definitions for the following terms:

- Normal form game.
- Strictly dominated strategy.
- Weakly dominated strategy.
- Best response strategy.
- Mixed strategy Nash equilibrium.

[5]

(b) State and prove a theorem giving a condition for which a strategy of the row player is a best response to a given strategy of the column player. [8]

(c) Consider the following Normal Form Game defined by:

$$M_r = \begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix} \qquad M_c = \begin{pmatrix} 2 & -2 \\ -3 & 2 \end{pmatrix}$$

State and justify which pairs of strategies are best responses to each other:

(i) $\sigma_r = (1, 0)$ and $\sigma_c = (0, 1)$

(ii) $\sigma_r = (1/5, 4/5)$ and $\sigma_c = (0, 1)$

(iii) $\sigma_r = (1/5, 4/5)$ and $\sigma_c = (1/2, 1/2)$

[9]

(d) Using your answer to (c) or otherwise, find all Nash equilibria for the game. [4]

2. Consider the donation game defined by:

$$M_r = \begin{pmatrix} b+c & c \\ b+2c & 2c \end{pmatrix} \qquad M_c = \begin{pmatrix} b+c & b+2c \\ c & 2c \end{pmatrix}$$

(a) Show that if $b > c > 0$ then this game is a Prisoner's Dilemma. [3]

(b) Obtain all Nash equilibrium for this game assuming the constraints of (a). [2]

(c) Consider a Moran Process on this game. Obtain an expression for the fixation probability of i mutants: playing the first strategy in a population of with N as a function of b, c and N.

You may use the following expression for the fixation probability in the general two type Moran process:

$$\rho_i = \frac{1 + \sum_{j=1}^{i-1} \prod_{k=1}^{j} \gamma_k}{1 + \sum_{j=1}^{N-1} \prod_{k=1}^{j} \gamma_k}$$

where:

$$\gamma_k = \frac{f_R(i)}{f_M(i)}$$

Where $f_R(i)$ and $f_M(i)$ is the fitness of a resident/mutant respectively in a population with i mutants. [8]

(d) Obtain the probability of a single mutant taking over for $N \in \{2, 3, 4\}$. [6]

(e) For $N = 4$ consider the limit as $b \to \infty$ and as $b \to c$. Comment on the implications of these results. [6]

3. (a) Define a characteristic function game $G = (N, v)$.

[2]

(b) Define the Shapley value.

[2]

(c) Obtain the Shapley value for the following characteristic function games:

$$v_1(c) = \begin{cases} 0, & \text{if } c = \emptyset \\ 8, & \text{if } c = \{1\} \\ 5, & \text{if } c = \{2\} \\ 9, & \text{if } c = \{3\} \\ 10, & \text{if } c = \{1,2\} \\ 11, & \text{if } c = \{2,3\} \\ 18, & \text{if } c = \{1,3\} \\ 30, & \text{if } c = \{1,2,3\} \end{cases} \qquad v_2(c) = \begin{cases} 0, & \text{if } c = \emptyset \\ 80, & \text{if } c = \{1\} \\ 10, & \text{if } c = \{2\} \\ 12, & \text{if } c = \{3\} \\ 80, & \text{if } c = \{1,2\} \\ 12, & \text{if } c = \{2,3\} \\ 80, & \text{if } c = \{1,3\} \\ 80, & \text{if } c = \{1,2,3\} \end{cases}$$

[8]

(d) Given a game $G = (N, v)$, a payoff vector λ satisfies the symmetry property if, for any pair of players i, j:
If $v(C \cup i) = v(C \cup j)$ for all coalitions $C \subseteq \Omega \setminus \{i, j\}$, then:

$$\lambda_i = \lambda_j$$

Prove that the Shapley value is efficient.

[6]

(e) The additivity property is:
Given two games $G_1 = (N, v_1)$ and $G_2 = (N, v_2)$, define their sum $G^+ = (N, v^+)$ by:

$$v^+(C) = v_1(C) + v_2(C) \quad \text{for all } C \subseteq \Omega$$

A payoff vector λ satisfies the additivity property if:

$$\lambda_i^{(G^+)} = \lambda_i^{(G_1)} + \lambda_i^{(G_2)}$$

Using the two games from part (c), demonstrate that the Shapley value has the additivity property.

[7]

4. (a) Define a **social welfare function**.

[2]

(b) State **Arrow's Impossibility Theorem**. Briefly discuss the implications of this theorem and ways in which the **Borda** or **Condorcet** methods respond to this impossibility.

[5]

(c) Consider the following preference profile over the set of alternatives $X = \{A, B, C\}$:

Number of voters	1st choice	2nd choice	3rd choice
4	A	B	C
3	B	C	A
2	C	A	B

(i) Construct the pairwise majority contests among the three alternatives.

(ii) Determine if there is a **Condorcet winner**.

(iii) Explain whether the collective preference relation is transitive.

[8]

(d) Apply the **Borda count** method to the same profile.

(i) Compute the Borda scores for each alternative.

(ii) Identify the Borda winner.

(iii) Does the Borda method select the same outcome as the Condorcet method?

[7]

(e) Define what it means for a voting rule to satisfy the **Independence of Irrelevant Alternatives (IIA)** property. Then, using the Borda count, give an example or explanation of how IIA may fail.

[3]