Tin Can Example - Writing & Mathematical Notation

QUESTION:

Consider a cylindrical tin can with radius r and height h. Let its volume V be non-zero and fixed. Find the relationship between h and r such that the surface area of the tin can is minimised.

SOLUTION:

First, let r be the radius of the cylinder, h be its height, and V be its volume. We have expressions for the tin's volume and surface area:

$$\begin{aligned} V &= \pi r^2 h \\ S &= 2\pi r^2 + 2\pi r h \end{aligned}$$

As the volume V is fixed, then we have a relationship between r and h that is dependant on V:

$$\begin{aligned} V &= \pi r^2 h \\ \frac{V}{\pi r^2} &= h \end{aligned}$$

Therefore the surface area now becomes a function of V and r:

$$\begin{aligned} S &= 2\pi r^2 + 2\pi r h \\ &= 2\pi r^2 + 2\pi r \left(\frac{V}{\pi r^2} \right) \\ &= 2\pi r^2 + \frac{2V}{r} \end{aligned}$$

The surface area is minimised when its derivative is equal to zero:

$$\begin{aligned} 0 &= \frac{dS}{dr} \\ &= 4\pi r - 2Vr^{-2} \end{aligned}$$

Then solving gives $\tilde{r}$, the value of r that minimises the surface area, in terms of V:

$$\begin{aligned} 4\pi\tilde{r} - \frac{2V}{\tilde{r}^2} &= 0 \\ 4\pi\tilde{r} &= \frac{2V}{\tilde{r}^2} \\ 4\pi\tilde{r}^3 &= 2V \\ \tilde{r}^3 &= \frac{V}{2\pi} \\ \tilde{r} &= \sqrt[3]{\frac{V}{2\pi}} \end{aligned}$$

However, as mentioned above, the volume V is fixed, and is itself a function of r and h. Substituting this in gives an implicit relationship between $\tilde{r}$ and h:

$$\begin{aligned} \tilde{r} &= \sqrt[3]{\frac{V}{2\pi}} \\ &= \sqrt[3]{\frac{\pi\tilde{r}^2 h}{2\pi}} \\ &= \sqrt[3]{\frac{\tilde{r}^2 h}{2}} \end{aligned}$$

Solving for either $\tilde{r}$ or h will give an explicit relationship between $\tilde{r}$ and h, as required.

$$
\begin{aligned}
\tilde{r} &= \sqrt[3]{\frac{\tilde{r}^2 h}{2}} \\
\tilde{r}^3 &= \frac{\tilde{r}^2 h}{2} \\
2\tilde{r}^3 &= \tilde{r}^2 h \\
2\tilde{r} &= h
\end{aligned}
$$

Therefore, for a fixed non-zero volume cylinder, the relationship between the radius r and height h that minimised the surface area is $h = 2r$.