4 (a) Provide definitions for the following terms:

- Normal form game.

  A N player **normal form game** consists of:
  - A finite set of N players;
  - Strategy spaces for the players: $S_1, S_2, S_3, \dots S_N$;
  - Payoff functions for the players: $u_i : S_1 \times S_2 \cdots \times S_N \to \mathbb{R}$

  [1]

- Strictly dominated strategy.

  In an N player normal form game. A pure strategy $s_i \in S_i$ is said to be **strictly dominated** if there is a strategy $\sigma_i \in \Delta S_i$ such that $u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$ of the other players.

  [1]

- Weakly dominated strategy.

  In an N player normal form game. A pure strategy $s_i \in S_i$ is said to be **weakly dominated** if there is a strategy $\sigma_i \in \Delta S_i$ such that $u_i(\sigma_i, s_{-i}) \geq u_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$ of the other players and there exists a strategy profile $\bar{s} \in S_{-i}$ such that $u_i(\sigma_i, \bar{s}) > u_i(s_i, s_{-i})$. [1]

- Best response strategy.

  In an N player normal form game. A strategy s^* for player i is a best response to some strategy profile s_{-i} if and only if $u_i(s^*, s_{-i}) \geq u_i(s, s_{-i})$ for all $s \in S_i$. [1]

- Nash equilibrium.

  In an N player normal form game. A Nash equilibrium is a strategy profile $\tau = (\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_N)$ such that:

  $$u_i(\tilde{s}) \geq u_i(\bar{s}_i, \tilde{s}_{-i}) \text{ for all } i$$

  [1]

For the remainder of this question consider the battle of the sexes game:

$$\begin{pmatrix} (3,2) & (0,0) \\ (1,1) & (2,3) \end{pmatrix}$$

(b) By clearly stating the techniques used, obtain all (if any) pure Nash equilibria.

By identifying best responses under the assumption of common knowledge of rationality we obtain r_1, s_1 and r_2, s_2 as Nash equilibria.

$$\begin{pmatrix} (\underline{3},\underline{2}) & (0,0) \\ (1,1) & (\underline{2},\underline{3}) \end{pmatrix}$$

[4]

(c) Plot the utilities to player 1 (the row player) assuming that the 2nd player (the column player) plays a mixed strategy: $\sigma_2 = (y, 1-y)$.

We have:

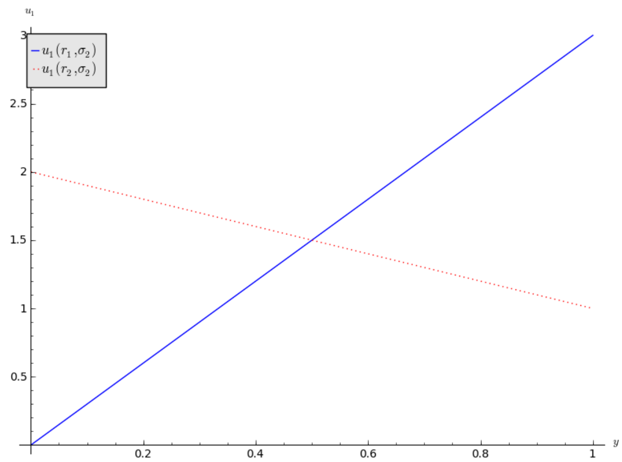
$$u_1(r_1, \sigma_2) = 3y$$

and

$$u_1(r_2, \sigma_2) = y + 2 - 2y = 2 - y$$

[1]

Which gives:



[1]

(d) Plot the utilities to player 2 (the column player) assuming that the 1st player (the row player) plays a mixed strategy: $\sigma_1 = (x, 1-x)$.
We have:

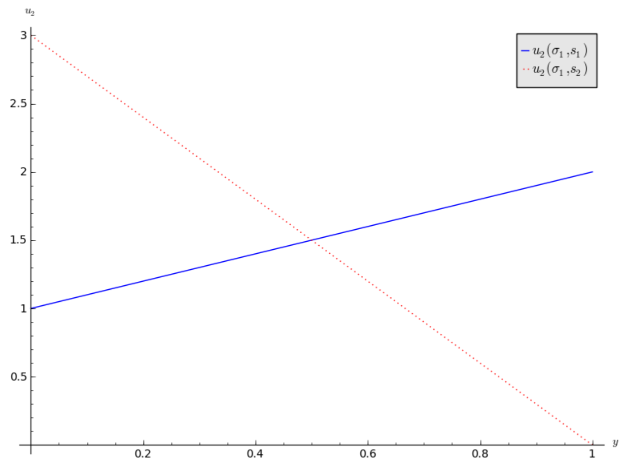
$$u_2(\sigma_1, s_1) = 2x + 1 - x = 1 + x$$

and

$$u_2(\sigma_1, s_2) = 3 - 3x$$

[1]

Which gives:



[1]

(e) Assuming that player 1 plays the mixed strategy $\sigma_1 = (x, 1-x)$, show that player 1's best response x^* to a mixed strategy $\sigma_2 = (y, 1-y)$ is given by:

$$x^* = \begin{cases} 0, & \text{if } y < 1/2 \\ 1, & \text{if } y > 1/2 \\ \text{indifferent}, & \text{otherwise} \end{cases}$$

We have $u_1(r_2, \sigma_2) = u_1(r_1, \sigma_2) \Rightarrow y = 1/2$. From the plots we see that if $y < 1/2$ then player 1's best response is to play r_2 which corresponds to $x = 0$, similarly for $y > 1/2$ and finally if $y = 1/2$ player 1 is indifferent.

[2]

Similarly show that player 2's best response y^* is given by:

$$y^* = \begin{cases} 0, & \text{if } x < 1/2 \\ 1, & \text{if } x > 1/2 \\ \text{indifferent}, & \text{otherwise} \end{cases}$$

We have $u_2(\sigma_1, s_1) = u_1(\sigma_1, s_2) \Rightarrow x = 1/2$. From the plots we see that if $x < 1/2$ then player 2's best response is to play s_2 which corresponds to $y = 0$, similarly for $x > 1/2$ and finally if $x = 1/2$ player 2 is indifferent.

[2]

(f) Use the above to obtain all Nash equilibria for the game.

We see that the only mixed strategy that is a pair of best responses is $(\sigma_1, \sigma_2) = ((1/2, 1/2), (1/2, 1/2))$. [2]

(g) Confirm this result by stating, proving and using the Equality of Payoffs theorem.

The equality of payoffs theorem states:

In an N player normal form game if the strategy profile (σ_i, s_{-i}) is a Nash equilibria then:

$$u_i(\sigma_i, s_{-i}) = u_i(s, s_{-i}) \text{ for all } s \in \mathcal{S}(\sigma_i) \text{ for all } 1 \leq i \leq N$$

[1]

Proof:

If $|\mathcal{S}(\sigma_i)| = 1$ then the proof is trivial.

We assume that $|\mathcal{S}(\sigma_i)| > 1$. Let us assume that the theorem is not true so that there exists $\bar{s} \in \mathcal{S}(\sigma)$ such that

$$u_i(\sigma_i, s_{-i}) \neq u_i(\bar{s}, s_{-i})$$

Without loss of generality let us assume that:

$$\bar{s} = \text{argmax}_{s \in \mathcal{S}(\sigma)} u_i(s, s_{-i})$$

Thus we have:

$$\begin{aligned} u_i(\sigma_i, s_{-i}) &= \sum_{s \in \mathcal{S}(\sigma_i)} \sigma_i(s) u(s, s_{-i}) \\ &\leq \sum_{s \in \mathcal{S}(\sigma_i)} \sigma_i(s) u(\bar{s}, s_{-i}) \\ &\leq u(\bar{s}, s_{-i}) \sum_{s \in \mathcal{S}(\sigma_i)} \sigma_i(s) \\ &\leq u(\bar{s}, s_{-i}) \end{aligned}$$

Giving:

$$u_i(\sigma_i, s_{-i}) < u_i(\bar{s}, s_{-i})$$

which implies that (σ_i, s_{-i}) is not a Nash equilibrium.

[4]

To verify the mixed Nash equilibria found previously we apply the theorem:

$$u_1(r_1, \sigma_2) = u_1(r_2, \sigma_2) \Rightarrow \tilde{y} = 1/2$$
$$u_2(\sigma_1, s_1) = u_2(\sigma_1, s_2) \Rightarrow \tilde{x} = 1/2$$

As required.

[1]

5 Consider the following stage game:

$$\begin{pmatrix}(2,2) & (5,0)\\(0,5) & (4,4)\end{pmatrix}$$

This game shall be referred to as the Prisoner's Dilemma. The first strategy for both players will be referred to as 'Cooperate' (C) and the second strategy will be referred to as 'Defect' (D). **Players aim to minimise their payoffs.**

Consider the following strategies:

- s_C: Always cooperate;
- s_D: Always defect;
- s_G: Start by cooperating until your opponent defects at which point defect in all future stages.

Assume $S_1 = S_2 = \{s_C, s_D, s_G\}$.

(a) Assuming a discounting factor of δ, obtain the utility to both players if the strategy pair (s_C, s_C) is played.

$$U_1(s_C, s_C) = U_2(s_C, s_C) = \sum_{t=1}^{\infty} \delta^{t-1} 2 = \frac{2}{1-\delta}$$

[2]

(b) Assuming a discounting factor of δ, obtain the utility to both players if the strategy pair (s_D, s_D) is played.

$$U_1(s_D, s_D) = U_2(s_D, s_D) = \sum_{t=1}^{\infty} \delta^{t-1} 4 = \frac{4}{1-\delta}$$

[2]

(c) For what values of δ is (s_G, s_G) a Nash equilibrium? **Recall that players aim to minimise their payoffs.**

Assuming both players play s_G, then $s_G = s_C$ and the outcome is $\frac{2}{1-\delta}$. The earlier deviation to s_D the more rewarding to the deviating player (all future rewards are more heavily discounted).

[1]

Assuming player 2 deviates at the first stage we have:

$$U_2(s_G, s_D) = 0 + \sum_{t=2}^{\infty} \delta^{t-1} 4 = \frac{4}{1-\delta} - 4$$

[2]

Recalling that players aim to minimise their utilities, this deviation is rational iff:

$$\frac{4}{1-\delta} - 4 < \frac{2}{1-\delta} \Rightarrow \delta < 1/2$$

[2]

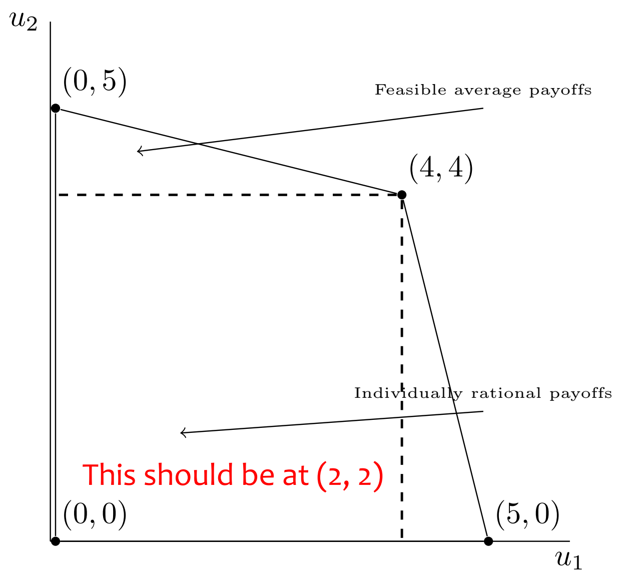
(d) Define the average payoff in an infinitely repeated game.

The average payoff is given by:

$$\frac{1}{\bar{T}} U_i(r,s) = (1-\delta)U_i(r,s)$$

[1]

(e) Plot the feasible average payoffs and the individually rational payoffs for the Prisoner's Dilemma. **Recall that players aim to minimise their payoffs.**



Error: The lower left point should in fact be at (2,2).

[4]

(f) Prove the following theorem (**for games where players aim to minimise their payoffs**):

"Let u_1^*, u_2^* be a pair of Nash equilibrium payoffs for a stage game. For every individually rational pair v_1, v_2 there exists $\bar{\delta}$ such that for all $1 > \bar{\delta} > \delta > 0$ there is a subgame perfect Nash equilibrium with payoffs v_1, v_2."

Let (σ_1^*, σ_2^*) be the stage Nash profile that yields (u_1^*, u_2^*). Now assume that playing $\bar{\sigma}_1 \in \Delta S_1$ and $\bar{\sigma}_2 \in \Delta S_2$ in every stage gives (v_1, v_2) (an individual rational payoff pair).

[1]

Consider the following strategy:

"Begin by using $\bar{\sigma}_i$ and continue to use $\bar{\sigma}_i$ as long as both players use the agreed strategies. If any player deviates: use σ_i^* for all future stages."

[1]

We begin by proving that the above is a Nash equilibrium.
Without loss of generality if player 1 deviates to $\sigma_1' \in \Delta S_1$ such that $u_1(\sigma_1', \bar{\sigma}_2) < v_1$ in stage k then:

$$U_1^{(k)} = (k-1)v_1 + u_1(\sigma_1', \bar{\sigma}_2) + u_1^* \left(\frac{1}{1-\delta} - \sum_{t=1}^{k} \delta^{t-1} \right)$$

[2]

Recalling that player 1 would receive v_1 in every stage with no deviation, the biggest gain to be made from deviating is if player 1 deviates in the first stage (all future gains are more heavily discounted). Thus if we can find $\bar{\delta}$ such that $\bar{\delta} > \delta$ implies that $U_1^{(1)} \geq \frac{v_1}{1-q}$ then player 1 has no incentive to deviate.

[2]

$$\begin{aligned} U_1^{(1)} = u_1(\sigma_1', \bar{\sigma}_2) + u_1^* \frac{\delta}{1-\delta} &\geq \frac{v_1}{1-\delta} \\ (1-\delta)u_1(\sigma_1', \bar{\sigma}_2) + u_1^*\delta &\geq v_1 \\ u_1(\sigma_1', \bar{\sigma}_2) - v_1 &\geq \delta(u_1(\sigma_1', \bar{\sigma}_2) - u_1^*) \end{aligned}$$

[2]

as $u_1(\sigma_1', \bar{\sigma}_2) < v_1 < u_1^*$, taking $\bar{\delta} = \frac{u_1(\sigma_1', \bar{\sigma}_2) - v_1}{u_1(\sigma_1', \bar{\sigma}_2) - u_1^*}$ gives the required required result for player 1 and repeating the argument for player 2 completes the proof of the fact that the prescribed strategy is a Nash equilibrium.

[1]

By construction this strategy is also a subgame perfect Nash equilibrium. Given any history **both** players will act in the same way and no player will have an incentive to deviate:

- If we consider a subgame just after any player has deviated from $\bar{\sigma}_i$ then both players use σ_i^*.
- If we consider a subgame just after no player has deviated from σ_i then both players continue to use $\bar{\sigma}_i$.

[2]

6 (a) Define a routing game (G, r, c).

A **routing game** (G, r, c) is defined on a graph $G = (V, E)$ with a defined set of sources s_i and sinks t_i. Each source-sink pair corresponds to a set of traffic (also called a commodity) r_i that must travel along the edges of G from s_i to t_i. Every edge e of G has associated to it a nonnegative, continuous and nondecreasing cost function (also called latency function) c_e. [2]

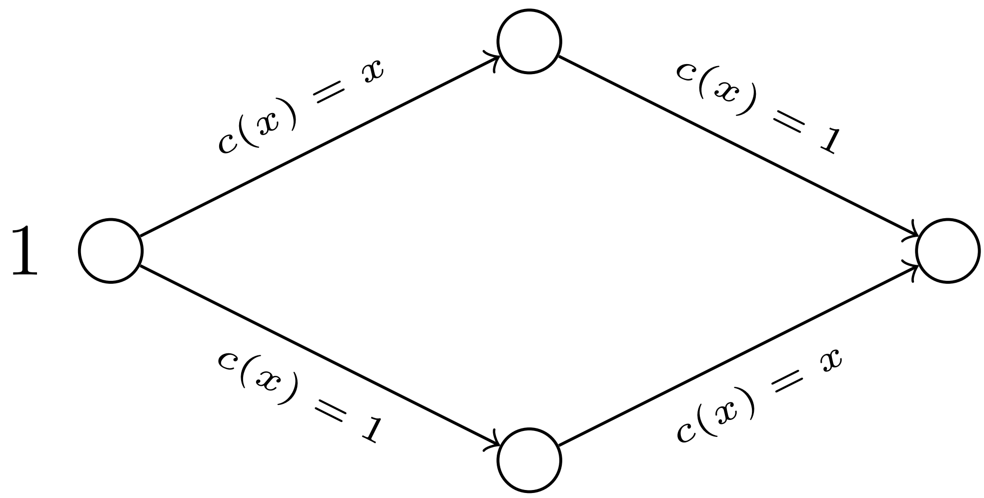
(b) Define a Nash flow and from first principles obtain the Nash flow for the following game:

For a routing game (G, r, c) a flow $\tilde{f}$ is called a **Nash flow** if and only if for every commodity i and any two paths $P_1, P_2 \in \mathcal{P}_i$ such that $f_{P_1} > 0$ then:

$$c_{P_1}(f) \leq c_{P_2}(f)$$

In other words a Nash flow ensures that all used paths have minimal costs.

[1]



For this game we see that if both paths are used then we must have:

$$x + 1 = (1 - x) + 1$$

[1]

(assuming x is the quantity of traffic using the top path)
thus the Nash flow is given by $\tilde{f} = (1/2, 1/2)$.

[1]

(c) Define an optimal flow and from first principles obtain the optimal flow for the above game.

For a routing game (G, r, c) we define the optimal flow f^* as the solution to the following optimisation problem:

Minimise $\sum_{e \in E} c_e(f_e)$:

Subject to:

$$\sum_{P \in \mathcal{P}_i} f_P = r_i \qquad \text{for all } i$$

$$f_e = \sum_{P \in \mathcal{P} \text{ if } e \in P} f_P \quad \text{for all } e \in E$$

$$f_P \geq 0$$

[2]

In our case $C(x) = x^2 + x + 1 - x + (1-x)^2 = 2x^2 - 2x + 2$, differentiating and equating to 0 we see that $f^* = (1/2, 1/2)$.

[1]

(d) State a theorem connecting the following function Φ to the Nash flow of a routing game:

$$\Phi(f) = \sum_{e \in E} \int_0^{f_e} c_e(x) dx$$

A feasible flow $\tilde{f}$ is a Nash flow for the routing game (G, r, c) if and only if it is a minima for $\Phi(f)$.

[2]

(e) Using the theorem from (d) confirm the Nash flow previously found in (b).

For our game we have: $\Phi(x) = x^2/2 + x + x + x^2/2 = x(1+x)$

Differentiating and equating to 0 gives $\tilde{f} = (1/2, 1/2)$ as required.

Error: Phi(x) = x^2/2+x+(1-x)^2/2+(1-x) [2]

(f) State a theorem connecting the marginal cost $c^*(x) = \frac{d(xc(x))}{dx}$ to the optimal flow of a routing game.

A feasible flow f^* is an optimal flow for (G, r, c) if and only if f^* is a Nash flow for (G, r, c^*).

[2]

(g) Using the theorem from (f) confirm the optimal flow previously found in (c) .

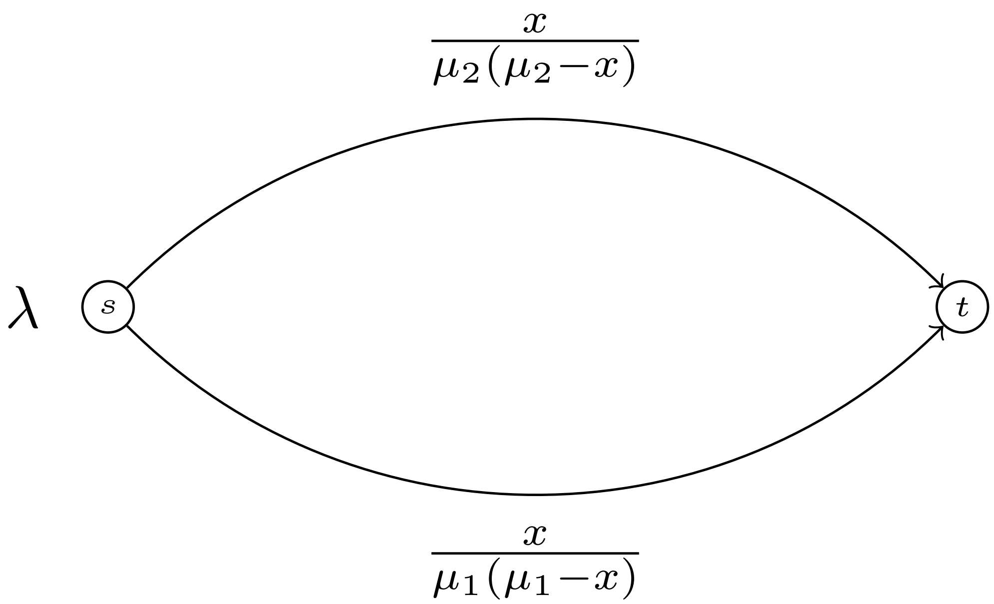
We have the cost of both paths: $2x + 1$ thus equating $2x + 1 = 2 - 2x + 1$ gives $f^* = (1/2, 1/2)$ as required.

[2]

(h) The expected time spent in an $M/M/1$ queue at steady state is given by:

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)}$$

Where μ, λ are the mean service and inter arrival rates and $\lambda < \mu$ respectively. Explain how a system with two $M/M/1$ queues and players choosing which queue to join can be studied using the following routing game:



The function W_q is nonnegative, continuous and nondecreasing in λ for $0 \le \lambda < \mu$. Thus using these functions as latency functions in λ assuming each $M/M/1$ queue has service rate parameter μ_i will give the described model. This holds since as long as the total arrival rate is less than the service rate at each queue. Finally, as a Markov process can be 'thinned' to another Markov process the steady state formulae will still hold. [2]

(i) Obtain the Nash and Optimal flows for the game in (h) with $\mu_1 = 4, \mu_2 = 3$ and $\lambda = 2$.

You might find it useful to know that the equation:

$$x^4 - 2x^3 + x^2 - 420x + 324 = 0$$

has a single solution in the range $0 \le x < 2$ given by $x \approx .772$.

For the Nash flow we need to equate both costs:

$$\frac{x}{3(3-x)} = \frac{2-x}{4(4-(2-x))}$$

which gives:

$$8x + 4x^2 - 18 + 6x + 9x - 3x^2 = x^2 + 23x - 18 = 0$$

which has non-negative root:

$$\frac{\sqrt{601} - 23}{2} \approx .7577$$

Thus the Nash flow is given by: $\tilde{f} = (.7577, 1.2423)$ (we see that the slow queue sees more players).

[3]

To obtain the Optimal flow we equate the marginal costs:

$$c^* = \frac{1}{dx}\frac{x^2}{\mu(\mu - x)} = \frac{2\,x}{(\mu - x)\mu} + \frac{x^2}{(\mu - x)^2\mu}$$

Thus we now equate both path costs:

$$\frac{2x}{3(3-x)} + \frac{x^2}{3(3-x)^2} = \frac{4-2x}{4(4-2+x)} + \frac{(2-x)^2}{4(4-2+x)^2}$$

which is equivalent to:

$$\frac{x^4 - 2\,x^3 + x^2 - 420\,x + 324}{12\,(x-3)^2(x+2)^2} = 0$$

Which implies (from the tip) $f^* = (.7715, 1.2285)$ (we see that the slow queue is slightly less busy).

[4]