1. Assume that a student can be in 1 of 4 states:

- Rich
- Average
- Poor
- In Debt

Assume the following transition probabilities:

- If a student is Rich, in the next time step the student will be:
  - Average: .75
  - Poor: .2
  - In Debt: .05

- If a student is Average, in the next time step the student will be:
  - Rich: .05
  - Average: .2
  - In Debt: .45

- If a student is Poor, in the next time step the student will be:
  - Average: .4
  - Poor: .3
  - In Debt: .2

- If a student is In Debt, in the next time step the student will be:
  - Average: .15
  - Poor: .3
  - In Debt: .55

Model the above as a discrete Markov chain and:

(a) Draw the corresponding Markov chain and obtain the corresponding stochastic matrix.
(b) Let us assume that a student starts their studies as “Average”. What will be the probability of them being “Rich” after 1, 2, 3 time steps?

\[
\pi^{(0)} = (0, 1, 0, 0)
\]

\[
\pi^{(1)} = \pi^{(0)} P = (0.05, 2, 3, 45)
\]

After 1 time step: 5% chance.

\[
\pi^{(2)} = \pi^{(0)} P^2 = (0.04, 2.65, 2.95, 4)
\]

After 2 time steps: 4% chance.

\[
\pi^{(3)} = \pi^{(0)} P^3 = (0.04275, 2.11, 2.96, 4.025)
\]

After 3 time step: 4.275% chance.

(c) What is the steady state probability vector associated with this Markov chain?

The linear system:

\[
\begin{align*}
0.05\pi_A + 0.1\pi_P & = \pi_R \\
0.75\pi_R + 0.2\pi_A + 0.4\pi_P + 0.15\pi_D & = \pi_A \\
0.2\pi_R + 0.3\pi_A + 0.3\pi_P + 0.3\pi_D & = \pi_P \\
0.05\pi_R + 0.45\pi_A + 0.2\pi_P + 0.55\pi_D & = \pi_D \\
\pi_R + \pi_A + \pi_P + \pi_D & = 1
\end{align*}
\]
2. Consider the following matrices. For the matrices that are stochastic matrices, draw the associated Markov Chain and obtain the steady state probabilities (if they exist, if not, explain why).

\[
\begin{align*}
\pi_R &= \frac{53}{1241} \\
\pi_A &= \frac{326}{1241} \\
\pi_P &= \frac{367}{1241} \\
\pi_D &= \frac{495}{1241}
\end{align*}
\]

(a) The Chain is given:

\[
\begin{bmatrix}
0 & 1 \\
0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
.5 & .25 & .25 \\
1 & 0 & 0 \\
0 & .23 & .77 \\
.8 & .1 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{1}{n} & \frac{n-1}{n} \\
\frac{n-1}{n} & \frac{1}{n}
\end{bmatrix}
\]

\[
\begin{bmatrix}
.2 & .3 & .5 \\
.1 & .1 & .8 \\
.7 & .1 & .2
\end{bmatrix}
\]

(b) Not a square matrix.

(c) The Chain is given:

\[
\begin{bmatrix}
.2 & .3 & .1 & .4 \\
0 & .3 & .7 & 0 \\
.5 & .2 & .1 & .2 \\
.1 & 0 & 0 & .9
\end{bmatrix}
\]

\[
\begin{bmatrix}
.2 & .3 & .5 \\
.3 & -.3 & 1 \\
.2 & .2 & .6 \\
.5 & 0 & 0 & .5
\end{bmatrix}
\]

\[
\begin{bmatrix}
\alpha & \beta \\
\omega & \gamma
\end{bmatrix}
\]

(d) The Chain is given:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
.5 & .1 & .1 \\
.8 & .1 & .7 \\
.1 & .8 & .1 \\
.2 & .2 & .6
\end{bmatrix}
\]
\[
\begin{align*}
\pi_1 + 2\pi_2 + 3\pi_3 &= \pi_1 \\
2\pi_1 + 3\pi_2 + 4\pi_3 &= \pi_2 \\
3\pi_1 + 4\pi_2 + 5\pi_3 &= \pi_3 \\
\pi_1 + 3\pi_2 + 5\pi_3 &= 1
\end{align*}
\]

gives:
\[
\pi = \left(\frac{32}{81}, \frac{29}{162}, \frac{23}{54}\right)
\]

(e) The Chain is given:

\[
\begin{align*}
2\pi_1 + 5\pi_2 + .1\pi_4 &= \pi_1 \\
3\pi_1 + 3\pi_2 + 2\pi_3 &= \pi_2 \\
4\pi_1 + 2\pi_2 + .9\pi_3 &= \pi_3 \\
\pi_1 + \pi_2 + \pi_3 + \pi_4 &= 1
\end{align*}
\]
has solution: \[ \pi = \left( \frac{49}{358}, \frac{29}{358}, \frac{14}{179}, \frac{126}{179} \right) \]

(f) \( P_{22} < 0 \)

(g) The Chain is given:

Immediate to see that: \[ \pi = (.25, .25, .25, .25) \]

(h) Only if \( \beta = 1 - \alpha > 0 \) and \( \omega = 1 - \gamma > 0 \).

3. Consider the following (incomplete) transition matrix:

\[
\begin{pmatrix}
? & 2 & 1.5 & .5 \\
? & -5 & 1 & 3 \\
5 & 2 & ? & 1 \\
1 & ? & 1 & -2
\end{pmatrix}
\]

(a) Fill in the missing values in the transition matrix.

\[
\begin{pmatrix}
-4 & 2 & 1.5 & .5 \\
1 & -5 & 1 & 3 \\
5 & 2 & -8 & 1 \\
1 & 0 & 1 & -2
\end{pmatrix}
\]

(b) Draw the Markov chain.
(c) Obtain the steady state probabilities.

\[
\begin{aligned}
-4\pi_1 + \pi_2 + 5\pi_3 + \pi_4 &= 0 \\
2\pi_1 - 5\pi_2 + 2\pi_3 &= 0 \\
1.5\pi_1 + \pi_2 - 8\pi_3 + \pi_4 &= 0 \\
.5\pi_1 + 3\pi_2 + \pi_3 - 2\pi_4 &= 0 \\
\pi_1 + \pi_2 + \pi_3 + \pi_4 &= 1
\end{aligned}
\]

has solution:

\[
\pi = \left(\frac{13}{43}, \frac{37}{215}, \frac{11}{86}, \frac{171}{430}\right)
\]

4. Consider the following matrices. For the matrices that are transition rate matrices, draw the associated Markov Chain and obtain the steady state probabilities (if they exist, if not, explain why).

(a) \[\begin{pmatrix}
-1 & 1 \\
-1 & 1
\end{pmatrix}\]

(b) \[\begin{pmatrix}
-5 & 0 & 5 \\
2 & 0 & -2 \\
0 & 3 & -3 \\
0 & 0 & 0
\end{pmatrix}\]

(c) \[\begin{pmatrix}
-3 & 3 & 0 \\
0 & -3 & 3 \\
3 & 0 & -3
\end{pmatrix}\]

(d) \[\begin{pmatrix}
-a & a & 0 \\
b & -(a+b) & a \\
0 & b & -b
\end{pmatrix}\]

(e) \[\begin{pmatrix}
-1 & 1 & 0 & 0 \\
0 & -4 & 2 & 2 \\
1 & 0 & -2 & 1 \\
2 & 0 & 0 & -2
\end{pmatrix}\]

(f) \[\begin{pmatrix}
0 & 0 & 0 \\
5 & -10 & 5 \\
10 & 0 & -10
\end{pmatrix}\]

(g) \[\begin{pmatrix}
-.5 & .5 & 0 & 0 \\
0 & -.5 & .5 & 0 \\
0 & 0 & -.5 & .5 \\
.5 & 0 & 0 & -.5
\end{pmatrix}\]

(h) \[\begin{pmatrix}
\alpha & \beta \\
\omega & \gamma
\end{pmatrix}\]

(a) \(P_{22} > 0\)

(b) Not a square matrix.
(c) The Chain is given:

\[
\pi = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right)
\]

(d) The Chain is given:

\[
\begin{align*}
-a\pi_1 + b\pi_2 &= 0 \\
b\pi_1 - (a + b)\pi_2 + b\pi_3 &= 0 \\
a\pi_2 - b\pi_3 &= 0
\end{align*}
\]

thus:

\[
\pi = \left( \pi_1, \frac{a}{b} \pi_1, \left( \frac{a}{b} \right)^2 \pi_1 \right)
\]

however:

\[
\sum_{i=1}^{3} \pi_i = 1
\]
so:

\[
\pi_1 = \frac{1}{1 + \frac{a}{b} + \left(\frac{a}{b}\right)^2}
\]

(e) The Chain is given:

\[
\begin{align*}
\pi_1 - \pi_3 + 2\pi_4 &= 0 \\
\pi_1 - 4\pi_2 &= 0 \\
2\pi_2 - 2\pi_3 &= 0 \\
2\pi_2 + \pi_3 - 2\pi_4 &= 0 \\
\pi_1 + \pi_2 + \pi_3 + \pi_4 &= 1
\end{align*}
\]
gives:

\[
\pi = \left(\frac{8}{15}, \frac{2}{15}, \frac{2}{15}, \frac{3}{15}\right)
\]

(f) The Chain is given:
\( \pi = (1, 0, 0) \)

(g) The Chain is given:

\[ \pi = \left( \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right) \]

(h) Only if \(-\alpha = \beta \geq 0\) and \(-\gamma = \omega \geq 0\)
\[
\begin{align*}
-\beta \pi_1 + \omega \pi_2 &= 0 \\
\pi_1 + \pi_2 &= 1
\end{align*}
\]

Gives:
\[
\pi = \left( \frac{\omega}{\beta + \omega}, \frac{\beta}{\beta + \omega} \right)
\]

if \(\beta + \omega = 0\) then no steady state exists.

5. Consider the following continuous Markov chain.

(a) Obtain the transition rate matrix.
\[
Q = \begin{pmatrix}
-1 & 0 & 1 & 0 \\
3 & -5 & 1 & 1 \\
2 & 0 & -2 & 0 \\
1 & 2 & 0 & -3
\end{pmatrix}
\]

(b) Obtain the steady state probabilities for this Markov chain.
\[
\begin{align*}
-\pi_1 + 3\pi_2 + 2\pi_3 + \pi_4 &= 0 \\
-5\pi_2 + 2\pi_4 &= 0 \\
\pi_1 + \pi_2 - 2\pi_3 &= 0 \\
\pi_2 - 3\pi_4 &= 0
\end{align*}
\]

has solution:
\[
\left( \frac{2}{3}, 0, \frac{1}{3}, 0 \right)
\]

(c) Obtain the corresponding discrete time Markov chain.
Taking \(\Delta t = \frac{1}{5}\) gives:
\[ P = \begin{pmatrix} \frac{4}{5} & 0 & \frac{1}{5} & 0 \\ \frac{1}{5} & \frac{1}{5} & 0 & \frac{2}{5} \\ 0 & \frac{1}{5} & \frac{1}{5} & 0 \\ \frac{2}{5} & 0 & \frac{2}{5} & \frac{1}{5} \end{pmatrix} \]

(d) Draw the corresponding Markov chain.

(e) Obtain the steady state probabilities for the discretized Markov chain.

\[ \begin{align*}
\frac{4}{5} \pi_1 + \frac{3}{5} \pi_2 + \frac{2}{5} \pi_3 + \frac{1}{5} \pi_4 &= \pi_1 \\
\frac{2}{5} \pi_4 &= \pi_2 \\
\frac{1}{5} \pi_1 + \frac{1}{5} \pi_2 + \frac{3}{5} \pi_3 &= \pi_3 \\
\frac{1}{5} \pi_2 + \frac{2}{5} \pi_4 &= \pi_4 \\
\pi_1 + \pi_2 + \pi_3 + \pi_4 &= 1
\end{align*} \]

has solution:

\[ \left( \frac{2}{3}, \frac{1}{3}, 0 \right) \]