1. (a) Provide definitions for the following terms:

- Normal form game.

  A N player **normal form game** consists of:
  - A finite set of N players;
  - Strategy spaces for the players: $S_1, S_2, S_3, \dots S_N$;
  - Payoff functions for the players: $u_i : S_1 \times S_2 \cdots \times S_N \to \mathbb{R}$

  [1]

- Strictly dominated strategy.

  In an N player normal form game. A pure strategy $s_i \in S_i$ is said to be **strictly dominated** if there is a strategy $\sigma_i \in \Delta S_i$ such that $u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$ of the other players.

  [1]

- Weakly dominated strategy.

  In an N player normal form game. A pure strategy $s_i \in S_i$ is said to be **weakly dominated** if there is a strategy $\sigma_i \in \Delta S_i$ such that $u_i(\sigma_i, s_{-i}) \geq u_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$ of the other players and there exists a strategy profile $\bar{s} \in S_{-i}$ such that $u_i(\sigma_i, \bar{s}) > u_i(s_i, s_{-i})$.

  [1]

- Best response strategy.

  In an N player normal form game. A strategy s^* for player i is a best response to some strategy profile s_{-i} if and only if $u_i(s^*, s_{-i}) \geq u_i(s, s_{-i})$ for all $s \in S_i$.
  [1]

- Nash equilibrium.

  In an N player normal form game. A Nash equilibrium is a strategy profile $\tau = (\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_N)$ such that:

  $$u_i(\tilde{s}) \geq u_i(\bar{s}_i, \tilde{s}_{-i}) \text{ for all } i$$

  [1]

(b) Consider the following game:

$$\begin{pmatrix} (2, \gamma) & (0, 3) \\ (0, 0) & (\gamma, 1) \end{pmatrix}$$

(i) Prove that a pure Nash equilibrium exists for all values of $\gamma \in \mathbb{R}$.
If $\gamma \leq 0$ then the best responses are given by:

$$\begin{pmatrix} (\underline{2}, \gamma) & (\underline{0}, \underline{3}) \\ (0, 0) & (\gamma, \underline{1}) \end{pmatrix}$$

[2]

If $3 \geq \gamma \geq 0$ then the best responses are given by:

$$\begin{pmatrix} (\underline{2}, \gamma) & (0, \underline{3}) \\ (0, 0) & (\underline{\gamma}, \underline{1}) \end{pmatrix}$$

So (r_2, c_2) is a pure Nash equilibrium.

[2]

If $3 \leq \gamma$ then the best responses are given by:

$$\begin{pmatrix} (\underline{2}, \underline{\gamma}) & (0, 3) \\ (0, 0) & (\underline{\gamma}, \underline{1}) \end{pmatrix}$$

So (r_1, c_1) and (r_2, c_2) are pure Nash equilibrium.

[3]

(ii) State the equality of payoffs theorem. Using this theorem obtain the value of γ (if it exists) for which the following (σ_1, σ_2) are mixed Nash equilibria for the game.

A. $(\sigma_1, \sigma_2) = ((1/2, 1/2), (1/4, 3/4))$
B. $(\sigma_1, \sigma_2) = ((1/4, 3/4), (1/2, 1/2))$
C. $(\sigma_1, \sigma_2) = ((1/5, 4/5), (1/6, 5/6))$

The equality of payoffs theorem states:
In an N player normal form game if the strategy profile (σ_i, s_{-i}) is a Nash equilibria then:

$$u_i(\sigma_i, s_{-i}) = u_i(s, s_{-i}) \text{ for all } s \in \mathcal{S}(\sigma_i) \text{ for all } 1 \leq i \leq N$$

[1]

A. $(\sigma_1, \sigma_2) = ((1/2, 1/2), (1/4, 3/4))$
If $((1/2, 1/2), (1/4, 3/4))$ is a Nash equilibrium the equality of payoffs theorem states:

$$u_2((1/2, 1/2), c_1) = u_2((1/2, 1/2), c_2)$$

which implies:

$$\gamma/2 = 2$$

[2]

So $\gamma = 4$. If $\gamma = 4$ then

$$u_1(r_1, (1/4, 3/4)) = 1/2 \quad u_1(r_2, (1/4, 3/4)) = 3$$

Thus by the equality of payoffs theorem this is not a Nash equilibrium.

[2]

B. $(\sigma_1, \sigma_2) = ((1/2, 1/2), (2/3, 1/3))$
If $((1/2, 1/2), (2/3, 1/1))$ is a Nash equilibrium the equality of payoffs theorem states:

$$u_2((1/2, 1/2), c_1) = u_2((1/2, 1/2), c_2)$$

which implies:

$$\gamma/2 = 2$$

[2]

So $\gamma = 4$. If $\gamma = 4$ then

$$u_1(r_1, (2/3, 1/3)) = 4/3 \quad u_1(r_2, (2/3, 1/3)) = 4/3$$

Thus by the equality of payoffs theorem this is a Nash equilibrium.

[2]

C. $(\sigma_1, \sigma_2) = ((1/5, 4/5), (7/9, 2/9))$
If $((1/5, 4/5), (7/9, 2/9))$ is a Nash equilibrium the equality of payoffs theorem states:

$$u_2((1/5, 4/5), c_1) = u_2((1/5, 4/5), c_2)$$

which implies:

$$\gamma = 7$$

[2]

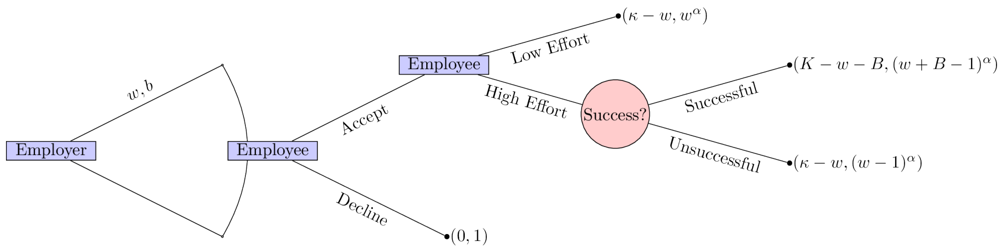
So $\gamma = 7$. If $\gamma = 7$ then

$$u_1(r_1, (7/9, 2/9)) = 14/9 \quad u_1(r_2, (1/4, 3/4)) = 14/9$$

Thus by the equality of payoffs theorem this is a Nash equilibrium.

[2]

2. (a) Give the extensive form representation for the above game.

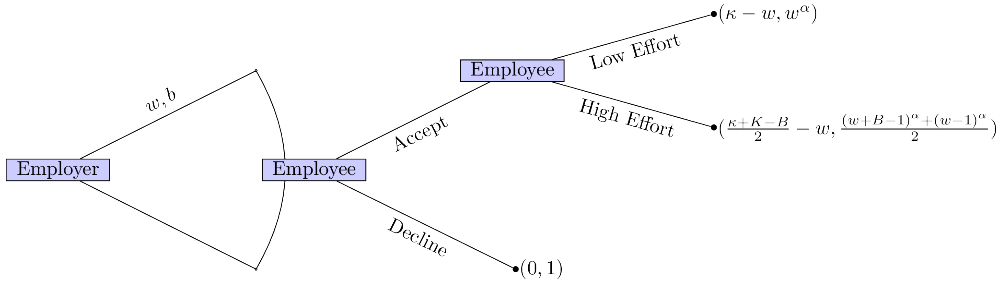


[5]

(b) List sensible assumptions with regards to the parameters and their interpretations.
Immediately we see that this game is trivial if $\kappa < \omega$ and $K < \omega + B$. Furthermore it seems sensible to only consider $K > \kappa$.

[4]

(c) Prove that the Nash equilibria for this game is the employer choosing $(w, B) = (1, 2^{1/\alpha})$ and the employee accepting the position.
We will solve this game using backward induction. The first step is shown.



[4]

If the employer would like a high level of effort he should set w, B such that:

$$\frac{(w + B - 1)^\alpha + (w - 1)^\alpha}{2} \geq w^\alpha$$

[4]

This ensures that the employee will put in a high level of effort. Furthermore it is in the employers interest to ensure that the employee accepts the job (we assume here that $\frac{\kappa + K + B}{2} \geq w$):

$$\frac{(w + B - 1)^\alpha + (w - 1)^\alpha}{2} \geq 1$$

[2]

This second inequality ensures that the employee accepts the position. Given that the position is accepted the employer would like to in fact minimise w, B thus we have:

$$\frac{(w+B-1)^\alpha+(w-1)^\alpha}{2}=1$$

Thus we have:

$$w^\alpha \leq 1$$

[2]

however we assume that $w \geq 1$ (so that the "wage is worth the effort") so we in fact have $w=1$. This then gives: $B=2^{1/\alpha}$. The expected utilities are then:

- Employer: $\frac{\kappa+K-2^{1/\alpha}}{2}-1$;
- Employee: 1.

[2]

(d) Explain how α affects the bonus offered by the employer.

The employer's utility is an increasing function in α. As the potential employee becomes more and more risk neutral the employer does not need to offer a high bonus to incite a high level of effort. [2]

3. (a) Define a matching game.

A matching game of size N is defined by two disjoint sets S and R or suitors and reviewers of size N. Associated to each element of S and R is a preference list:

$$f : S \to R^N \text{ and } g : R \to S^N$$

A matching M is a any bijection between S and R. If $s \in S$ and $r \in R$ are matched by M we denote:

$$M(s) = r$$

[2]

(b) Define a blocking pair for a matching game.

A pair (s, r) is said to **block** a matching M if $M(s) \neq r$ but s prefers r to $M(s)$ and r prefers s to $M^{-1}(r)$.

[2]

(c) Define a stable matching for a matching game.

A matching M with no blocking pair is said to be stable.

[2]

(d) Consider the following algorithm:

(i) Assign every $s \in S$ and $r \in R$ to be unmatched

(ii) Pick some unmatched $s \in S$, let r be the top of s's preference list:

- If r is unmatched set $M(s) = r$
- If r is matched:
  - If r prefers s to $M^{-1}(r)$ then set $M(r) = s$
  - Otherwise s remains unmatched and remove r from s's preference list.

(iii) Repeat step 2 until all $s \in S$ are matched.

Prove that all possible executions of this algorithm yield the same stable matching and in this stable matching every suitor has the best possible partner in any stable matching.

Suppose that an arbitrary execution α of the algorithm gives M and that another execution β gives M' such that $\exists\, s \in S$ such that s prefers $r' = M'(s)$ to $r = M(s)$.

[3]

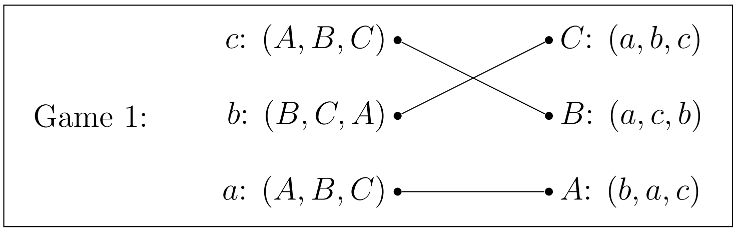
Without loss of generality this implies that during α r' must have rejected s. Suppose, again without loss of generality that this was the first occasion that a rejection occured during α and assume that this rejection occurred because $r' = M(s')$. This implies that s' has no stable match that is higher in s''s preference list than r' (as we have assumed that this is the first rejection).

[3]

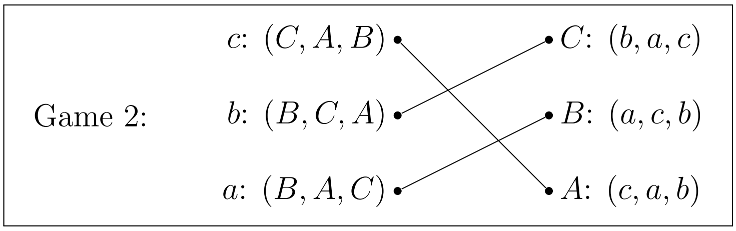
Thus s' prefers r' to $M'(s')$ so that (s', r') blocks M'. Each suitor is therefore matched in M with his favorite stable reviewer and since α was arbitrary it follows that all possible executions give the same matching.

[3]

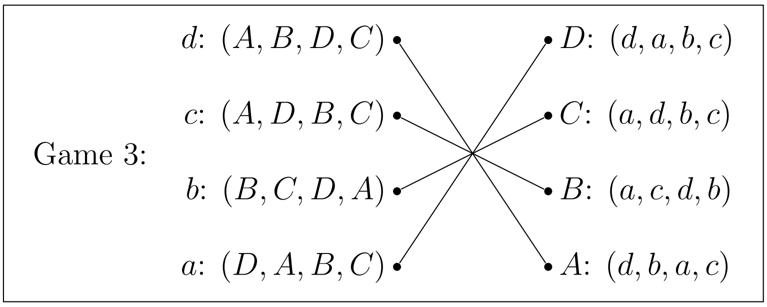
(e) Obtain stable matchings for the following games:



[3]



[3]



[4]