Queueing Theory Exercise Sheet Solutions

1. Fill in the gaps in the following table:

Statistic	Notation	$M/M/1$	$M/M/2$	$M/M/k$
Number of people in queue	L_q	$\frac{\rho^2}{1-\rho}$	$\frac{2\rho^3}{1-\rho^2}$	$\frac{\left(\frac{\lambda}{\mu}\right)^{k+1}\pi_0}{kk!\left(1-\frac{\lambda}{k\mu}\right)^2}$
Number of people in system	L_c	$\frac{\rho}{1-\rho}$	$\frac{2\rho}{1-\rho^2}$	$\frac{\left(\frac{\lambda}{\mu}\right)^{k+1}\pi_0}{kk!\left(1-\frac{\lambda}{k\mu}\right)^2}+\frac{\lambda}{\mu}$
Average waiting time in queue	W_q	$\frac{\rho}{\mu(1-\rho)}$	$\frac{\rho^2}{\mu(1-\rho^2)}$	$\frac{\left(\frac{\lambda}{\mu}\right)^{k}\pi_0}{kk!\left(1-\frac{\lambda}{k\mu}\right)^2\mu}$
Average time in system	W_c	$\frac{1}{\mu(1-\rho)}$	$\frac{1}{\mu(1-\rho^2)}$	$\frac{\left(\frac{\lambda}{\mu}\right)^{k}\pi_0}{kk!\left(1-\frac{\lambda}{k\mu}\right)^2\mu}+\frac{1}{\mu}$

2. • FIFO:

$$\begin{aligned}
\text{Total waiting time} &= 0+1+(1+2)+(1+2+3)+\cdots+(1+2+3+\cdots+(n-1))\\
&= \textstyle\sum_{k=1}^{n-1}\sum_{j=0}^{k} j = \sum_{k=1}^{n-1}\frac{k(k+1)}{2}\\
&= \textstyle\frac{1}{2}\left(\sum_{k=1}^{n-1}k^2+\sum_{k=1}^{n-1}k\right)\\
&= \textstyle\frac{1}{2}\left(\frac{(n-1)n(2n-1)}{6}+\frac{n(n-1)}{2}\right)\\
&= \textstyle\frac{1}{2}\left(\frac{(n-1)n(2n+2)}{6}\right)=\frac{(n-1)n(n+1)}{6}
\end{aligned}$$

However a total of n customers are served thus:

$$W_q=\frac{(n-1)(n+1)}{6}=\frac{n^2-1}{6}$$

as required.

• LIFO

$$\begin{aligned}
\text{Total waiting time} &= 0+n+(n+(n-1))+\cdots+(n+\cdots+2)\\
&= \textstyle\sum_{k=0}^{n-2}\sum_{j=0}^{k}(n-j)=\sum_{k=0}^{n-2}\sum_{j=n-k}^{n}j=\sum_{k=0}^{n-2}\left(\sum_{j=0}^{n}j-\sum_{j=0}^{n-k-1}j\right)\\
&= \textstyle\sum_{k=0}^{n-2}\left(\frac{n(n+1)}{2}-\frac{(k-n)(1+k-n)}{2}\right)=\sum_{k=0}^{n-2}\frac{(k+1)(2n-k)}{2}\\
&= \textstyle\frac{1}{2}\left(-\sum_{k=0}^{n-2}k^2+(2n-1)\sum_{k=0}^{n-2}k+\sum_{k=0}^{n-2}2n\right)\\
&= \textstyle\frac{1}{2}\left(-\frac{(n-2)(n-1)(2n-3)}{6}+\frac{(n-2)(n-1)(2n-1)}{2}+2n(n-1)\right)=\frac{(n-1)n(n+1)}{3}
\end{aligned}$$

However a total of n customers are served thus:

$$W_q=\frac{(n-1)(n+1)}{3}=\frac{n^2-1}{3}$$

as required.

3. We have:

$$\begin{aligned} E(T) &= \sum_{n=0}^{\infty} \frac{n(\lambda t)^n e^{-\lambda t}}{n!} \\ &= e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{(n-1)!} \\ &= \lambda t e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!} \\ &= \lambda t e^{-\lambda t} e^{\lambda t} \end{aligned}$$

as required.

5 minutes $\Leftrightarrow \frac{1}{12}$ hours. Thus, $\lambda t = \frac{24}{12} = 2$.

$$\begin{aligned} P(X=0) &= e^{-2} \approx .135335 \\ P(X=1) &= \frac{2e^{-2}}{1} \approx .270671 \\ P(X=2) &= \frac{2^2 e^{-2}}{2} \approx .270671 \\ P(X=3) &= \frac{2^3 e^{-2}}{6} \approx .180447 \end{aligned}$$

We have:

$P(X \geq 4) = 1 - P(X \leq 3) = 1 - P(X=3) - P(X=2) - P(X=1) - P(X=0) \approx .142877$

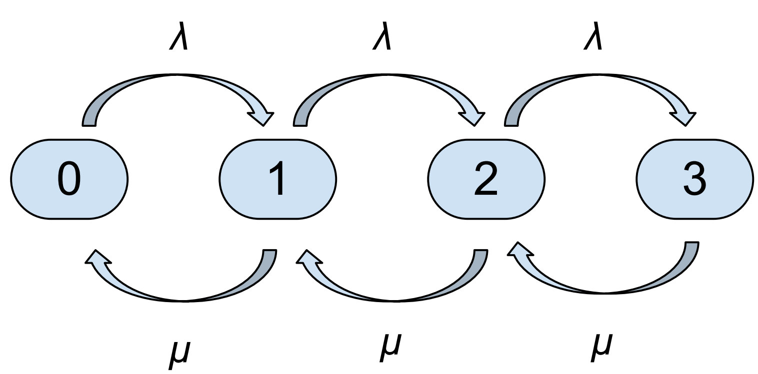
4. We have:

$$\begin{aligned} E(T) &= \int_0^{\infty} \lambda t e^{-\lambda t} dt \\ &= -\frac{(\lambda t)e^{-\lambda t}}{\lambda}\Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda t} dt \\ &= -\frac{e^{-\lambda t}}{\lambda}\Big|_0^{\infty} = \frac{1}{\lambda} \end{aligned}$$

as required.
By definition: $F(x) = P(X \leq x) = \int_0^x \lambda e^{-\lambda t} dt = 1 - e^{-\lambda x}$. We have $\lambda = 36$ which gives:

$$\begin{aligned} P(X \leq \tfrac{1}{60}) &= 1 - e^{\frac{-36}{60}} \approx .451188 \\ P(X \leq \tfrac{1}{30}) &= 1 - e^{\frac{-36}{30}} \approx .698806 \\ P(X > \tfrac{1}{30}) &= 1 - P(X \leq \tfrac{1}{30}) = e^{\frac{-36}{30}} \approx .301194 \end{aligned}$$

5. The Markov chain is given:



which has rate matrix:

$$\begin{pmatrix} -\lambda & \lambda & 0 & 0 \\ \mu & -(\lambda+\mu) & \lambda & 0 \\ 0 & \mu & -(\lambda+\mu) & \lambda \\ 0 & 0 & \mu & -\mu \end{pmatrix}$$

The steady state equations are given by:

$$\begin{aligned} \pi_0\lambda &= \pi_1\mu \\ \pi_1(\lambda+\mu) &= \pi_0\lambda + \pi_2\mu \\ \pi_2(\lambda+\mu) &= \pi_1\lambda + \pi_3\mu \\ \pi_3\mu &= \pi_2\lambda \end{aligned}$$

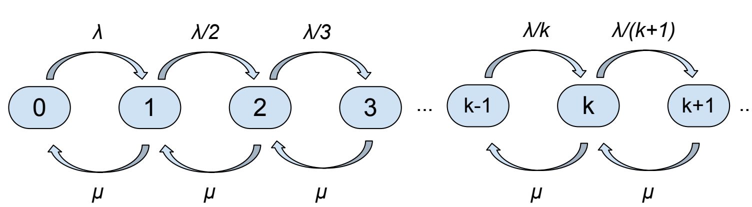
The solution for this system can be found (make sure you are able to do this!) to be:

$$\begin{aligned} \pi_0 &= \tfrac{1}{1+\rho+\rho^2+\rho^3} \\ \pi_1 &= \rho\pi_0 \\ \pi_2 &= \rho^2\pi_0 \\ \pi_3 &= \rho^3\pi_0 \end{aligned}$$

as required. The mean number of vehicles at the station is given by:

$$\begin{aligned} \textstyle\sum_{i=0}^{3} i\pi_i &= \textstyle\sum_{i=0}^{3} \frac{i\rho^i}{1+\rho+\rho^2+\rho^3} \\ &= \tfrac{1}{1+\rho+\rho^2+\rho^3} \textstyle\sum_{i=0}^{3} i\rho^i \\ &= \tfrac{\rho(1+2\rho+3\rho^2)}{1+\rho+\rho^2+\rho^3} \end{aligned}$$

6. We have the Markov chain given by



The steady state equations are:

$$\begin{aligned} \pi_0\lambda &= \pi_1\mu \\ \pi_1(\tfrac{\lambda}{2}+\mu) &= \pi_0\lambda + \pi_2\mu \\ &\vdots \\ \pi_k(\tfrac{\lambda}{k+1}+\mu) &= \pi_{k-1}\tfrac{\lambda}{k} + \pi_{k+1}\mu \end{aligned}$$

By inspection we have:

$$\pi_1 = \rho\pi_0$$
$$\pi_2 = \tfrac{\rho^2}{2}\pi_0$$
$$\pi_3 = \tfrac{\rho^3}{3!}\pi_0$$
$$\vdots$$
$$\pi_{k+1} = \tfrac{\rho^{k+1}}{(k+1)!}\pi_0$$

We conjecture that $\pi_i = \frac{\rho^i}{i!}\pi_0$ for all $i \geq 1$. We prove this by induction. For $i = 1$ we have $\pi_1 = \rho\pi_0$ as required. Let us now assume that $\pi_i = \frac{\rho^i}{i!}\pi_0$ for all $i \leq n$ for some $n \geq 1$. From above we then have:

$$\pi_{n+1} = \frac{\pi_n(\frac{\lambda}{n+1} + \mu) - \pi_{n-1}\frac{\lambda}{n}}{\mu} = \pi_0\left(\frac{\rho^n}{n!}\left(\frac{\rho}{n+1} + 1\right) - \frac{\rho}{n!}\right) = \frac{\rho^{n+1}}{(n+1)!}\pi_0$$

as required.

Finally, taking the sum or probabilities equal to 1, we have:

$$\sum_{k=0}^{\infty}\frac{\rho^k}{k!}\pi_0 = 1 \Rightarrow \pi_0 = e^{-\rho}$$

thus we have $\pi_i = \frac{\rho^i}{i!}e^{-\rho}$ for all $i \geq 0$.

7. We have $\lambda = 5$ and $\mu = 6$, thus for the formula for the $M/M/1$ queue we have: $L_c = \frac{\rho}{1-\rho} = 5$ which gives an hourly cost of $5 + 5 \times 8 = \$45$ per hour.

   If a second distribution centre is setup we can expect the arrival rate at each centre to be $\lambda = \frac{5}{2}$. We still have $\mu = 6$. The average number of workers at each centre is: $L_c = \frac{5}{7}$, thus $\frac{10}{7}$ overall. This gives an hourly cost of $2 \times 5 + \frac{80}{7} \approx \$21.43$ per hour. Thus, employing a second distributor is justified.

8. We can use the formulas from question 1 to obtain the following table:

	$M/M/1$	$M/M/2$
λ	$.4$	$.4$
μ	$.8$	$.5$
ρ	$.5$	$.4$
π_0	$(1-\rho) = .5$	$\frac{1-\rho}{1+\rho} = \frac{6}{14} \approx .4286$
L_q	$\frac{(.5)^2}{1-.5}.5$	$\frac{2*(.4)^3}{1-.4^2} \approx .15$
W_q	$\frac{5}{4} = 1.25$	$\frac{8}{21} \approx .38095$
W_c	$\frac{5}{2} = 2.5$	$\frac{50}{21} \approx 2.38095$
L_c	1	$\frac{20}{21} \approx .952381$
$P(\text{wait})$	$1 - \pi_0 = \frac{1}{2}$	$1 - \pi_0 - \pi_1 = 1 - \frac{6}{14}(1 + \frac{4}{5}) \approx .228570$

   (The last point uses the fact that $\pi_1 = \frac{\lambda}{\mu}\pi_0$ in an $M/M/2$ queue.)

   From this analysis it could be recommended that the new proposal is implemented. Indeed, this would give a shorter wait to customers (W_q and $P(\text{wait})$).