Markov Chains Exercise Sheet - Solutions

Last updated: October 17, 2012.

1. Assume that a student can be in 1 of 4 states:

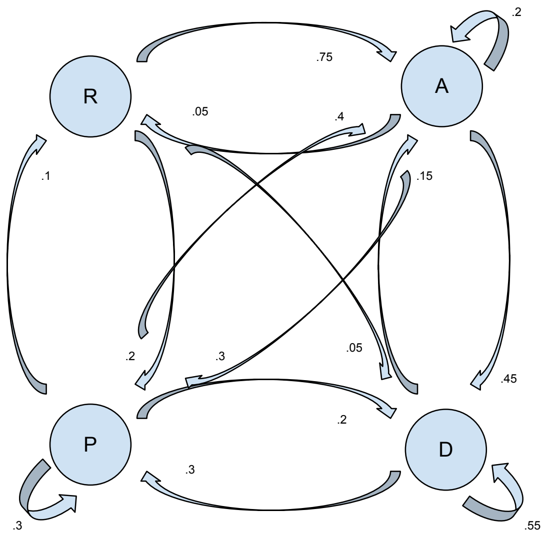
    - Rich
    - Average
    - Poor
    - In Debt

    Assume the following transition probabilities:

    - If a student is Rich, in the next time step the student will be:
        - Average: .75
        - Poor: .2
        - In Debt: .05
    - If a student is Average, in the next time step the student will be:
        - Rich: .05
        - Average: .2
        - In Debt: .45
    - If a student is Poor, in the next time step the student will be:
        - Average: .4
        - Poor: .3
        - In Debt: .2
    - If a student is In Debt, in the next time step the student will be:
        - Average: .15
        - Poor: .3
        - In Debt: .55

    Model the above as a discrete Markov chain and:

    (a) Draw the corresponding Markov chain and obtain the corresponding stochastic matrix.



$$P = \begin{pmatrix} 0 & .75 & .2 & .05 \\ .05 & .2 & .3 & .45 \\ .1 & .4 & .3 & .2 \\ 0 & .15 & .3 & .55 \end{pmatrix}$$

(b) Let us assume that a student starts their studies as "Average". What will be the probability of them being "Rich" after 1,2,3 time steps?

$$\pi^{(0)} = (0, 1, 0, 0)$$

$$\pi^{(1)} = \pi^{(0)}P = (.05, .2, .3, .45)$$

After 1 time step: 5% chance.

$$\pi^{(2)} = \pi^{(0)}P^2 = (.04, .265, .295, .4)$$

After 2 time steps: 4% chance.

$$\pi^{(3)} = \pi^{(0)}P^3 = (.04275, .211, .296, .4025)$$

After 3 time step: 4.275% chance.

(c) What is the steady state probability vector associated with this Markov chain?

The linear system:

$$\begin{cases} .05\pi_A + .1\pi_P & = \pi_R \\ .75\pi_R + .2\pi_A + .4\pi_P + .15\pi_D & = \pi_A \\ .2\pi_R + .3\pi_A + .3\pi_P + .3\pi_D & = \pi_P \\ .05\pi_R + .45\pi_A + .2\pi_P + .55\pi_D & = \pi_D \\ \pi_R + \pi_A + \pi_P + \pi_D = 1 \end{cases}$$

has solution:

$$\begin{cases} \pi_R = \frac{53}{1241} \\ \pi_A = \frac{326}{1241} \\ \pi_P = \frac{367}{1241} \\ \pi_D = \frac{495}{1241} \end{cases}$$

2. Consider the following matrices. For the matrices that are stochastic matrices, draw the associated Markov Chain and obtain the steady state probabilities (if they exist, if not, explain why).

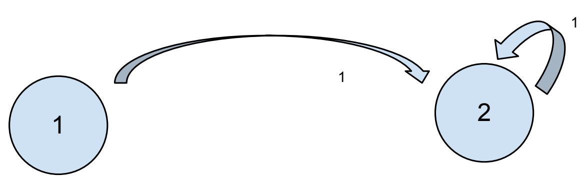
$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} .5 & .25 & .25 \\ 1 & 0 & 0 \\ 0 & .23 & .77 \\ .8 & .1 & .1 \end{pmatrix} \quad \begin{pmatrix} \frac{1}{n} & \frac{n-1}{n} \\ \frac{n-1}{n} & \frac{1}{n} \end{pmatrix} \quad \begin{pmatrix} .2 & .3 & .5 \\ .1 & .1 & .8 \\ .7 & .1 & .2 \end{pmatrix}$$

(a) (b) (c) (d)

$$\begin{pmatrix} .2 & .3 & .1 & .4 \\ 0 & .3 & .7 & 0 \\ .5 & .2 & .1 & .2 \\ .1 & 0 & 0 & .9 \end{pmatrix} \quad \begin{pmatrix} .2 & .3 & .5 \\ .3 & -.3 & 1 \\ .2 & .2 & .6 \end{pmatrix} \quad \begin{pmatrix} .5 & .5 & 0 & 0 \\ 0 & .5 & .5 & 0 \\ 0 & 0 & .5 & .5 \\ .5 & 0 & 0 & .5 \end{pmatrix} \quad \begin{pmatrix} \alpha & \beta \\ \omega & \gamma \end{pmatrix}$$

(e) (f) (g) (h)

(a) The Chain is given:

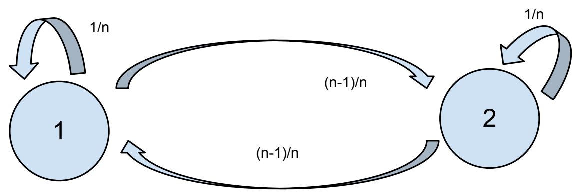


$$\pi = (0, 1)$$

(b) Not a square matrix.

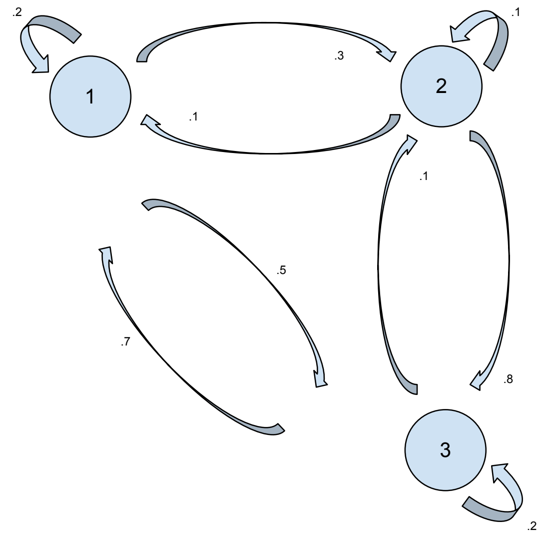
(c) The Chain is given:

For $0 < n$:



$$\pi = (.5, .5)$$

(d) The Chain is given:

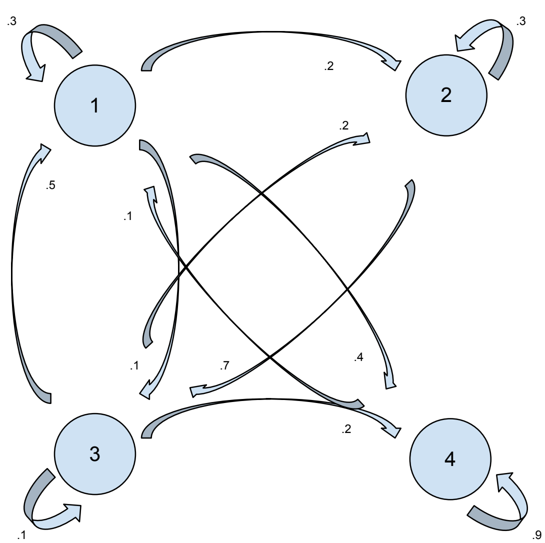


$$\begin{cases} .2\pi_1 + .1\pi_2 + .7\pi_3 & = \pi_1 \\ .3\pi_1 + .1\pi_2 + .1\pi_3 & = \pi_2 \\ .5\pi_1 + .8\pi_2 + .2\pi_3 & = \pi_3 \\ \pi_1 + \pi_2 + \pi_3 & = 1 \end{cases}$$

gives:

$$\pi = \left(\frac{32}{81}, \frac{29}{162}, \frac{23}{54}\right)$$

(e) The Chain is given:



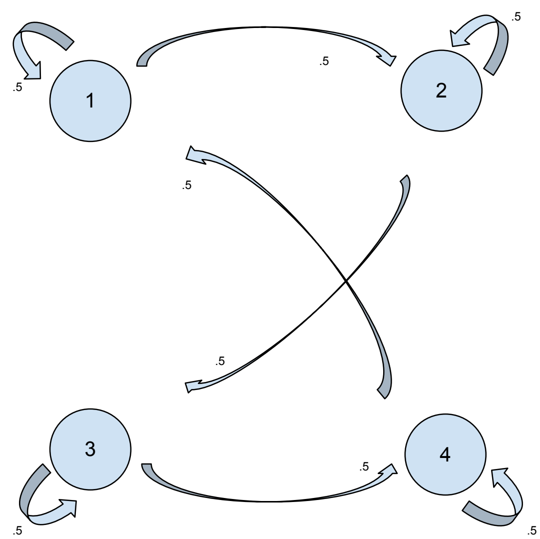
$$\begin{cases} .2\pi_1 + .5\pi_2 + .1\pi_4 & = \pi_1 \\ .3\pi_1 + .3\pi_2 + .2\pi_3 & = \pi_2 \\ .1\pi_1 + .7\pi_2 + .1\pi_3 & = \pi_3 \\ .4\pi_1 + .2\pi_3 + .9\pi_4 & = \pi_4 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 & = 1 \end{cases}$$

has solution:

$$\pi = \left(\frac{49}{358}, \frac{29}{358}, \frac{14}{179}, \frac{126}{179}\right)$$

(f) $P_{22} < 0$

(g) The Chain is given:



Immediate to see that:

$$\pi = (.25, .25, .25, .25)$$

(h) Only if $\beta = 1 - \alpha > 0$ and $\omega = 1 - \gamma > 0$.

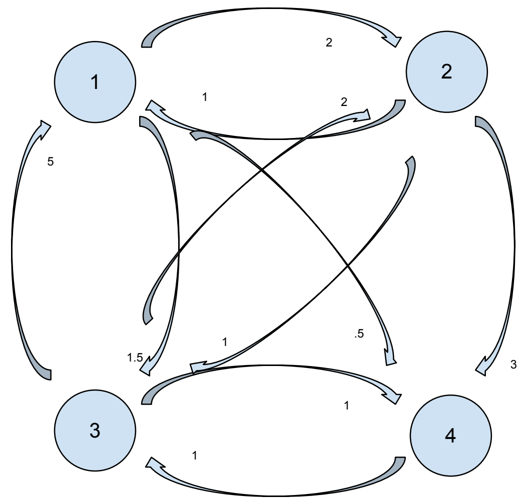
3. Consider the following (incomplete) transition matrix:

$$\begin{pmatrix} ? & 2 & 1.5 & .5 \\ ? & -5 & 1 & 3 \\ 5 & 2 & ? & 1 \\ 1 & ? & 1 & -2 \end{pmatrix}$$

(a) Fill in the missing values in the transition matrix.

$$\begin{pmatrix} -4 & 2 & 1.5 & .5 \\ 1 & -5 & 1 & 3 \\ 5 & 2 & -8 & 1 \\ 1 & 0 & 1 & -2 \end{pmatrix}$$

(b) Draw the Markov chain.



(c) Obtain the steady state probabilities.

$$\begin{cases} -4\pi_1 + \pi_2 + 5\pi_3 + \pi_4 &= 0 \\ 2\pi_1 - 5\pi_2 + 2\pi_3 &= 0 \\ 1.5\pi_1 + \pi_2 - 8\pi_3 + \pi_4 &= 0 \\ .5\pi_1 + 3\pi_2 + \pi_3 - 2\pi_4 &= 0 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 &= 1 \end{cases}$$

has solution:

$$\pi = \left(\frac{13}{43}, \frac{37}{215}, \frac{11}{86}, \frac{171}{430}\right)$$

4. Consider the following matrices. For the matrices that are transition rate matrices, draw the associated Markov Chain and obtain the steady state probabilities (if they exist, if not, explain why).

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$$
(a)

$$\begin{pmatrix} -5 & 0 & 5 \\ 2 & 0 & -2 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$
(b)

$$\begin{pmatrix} -3 & 3 & 0 \\ 0 & -3 & 3 \\ 3 & 0 & -3 \end{pmatrix}$$
(c)

$$\begin{pmatrix} -a & a & 0 \\ b & -(a+b) & a \\ 0 & b & -b \end{pmatrix}$$
(d)

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -4 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & -2 \end{pmatrix}$$
(e)

$$\begin{pmatrix} 0 & 0 & 0 \\ 5 & -10 & 5 \\ 10 & 0 & -10 \end{pmatrix}$$
(f)

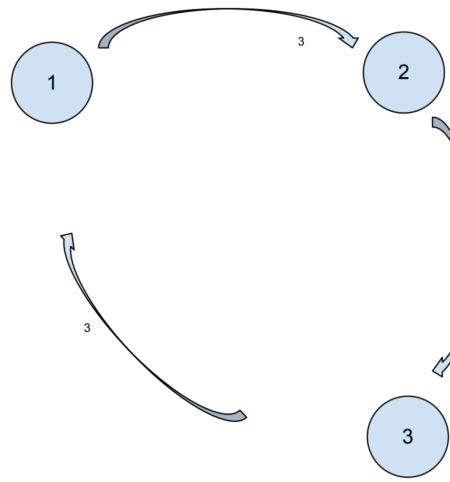
$$\begin{pmatrix} -.5 & .5 & 0 & 0 \\ 0 & -.5 & .5 & 0 \\ 0 & 0 & -.5 & .5 \\ .5 & 0 & 0 & -.5 \end{pmatrix}$$
(g)

$$\begin{pmatrix} \alpha & \beta \\ \omega & \gamma \end{pmatrix}$$
(h)

(a) $P_{22} > 0$

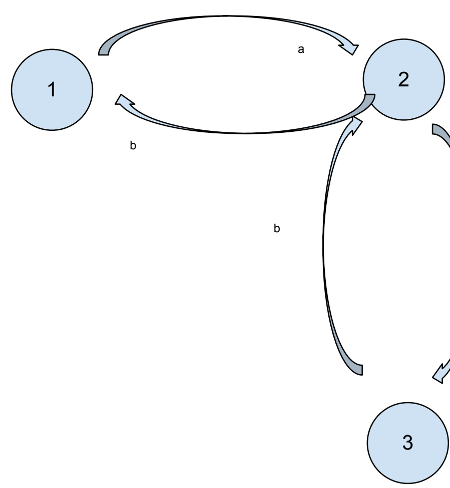
(b) Not a square matrix.

(c) The Chain is given:



$$\pi = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

(d) The Chain is given:



$$\begin{cases} -a\pi_1 + b\pi_2 & = 0 \\ b\pi_1 - (a+b)\pi_2 + b\pi_3 & = 0 \\ a\pi_2 - b\pi_3 & = 0 \end{cases}$$

thus:

$$\pi = \left(\pi_1, \frac{a}{b}\pi_1, \left(\frac{a}{b}\right)^2 \pi_1\right)$$

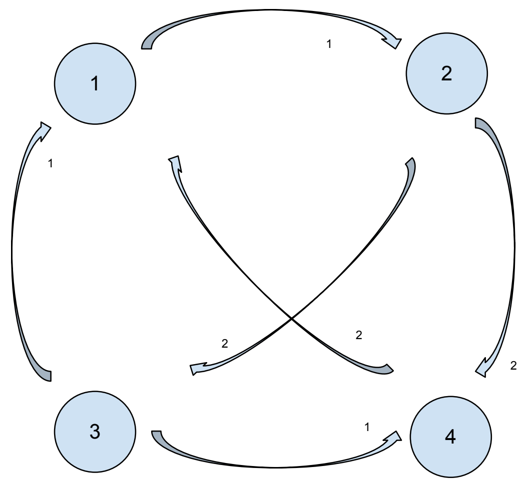
however:

$$\sum_{i=1}^{3} \pi_i = 1$$

so:

$$\pi_1 = \frac{1}{1 + \frac{a}{b} + \left(\frac{a}{b}\right)^2}$$

(e) The Chain is given:

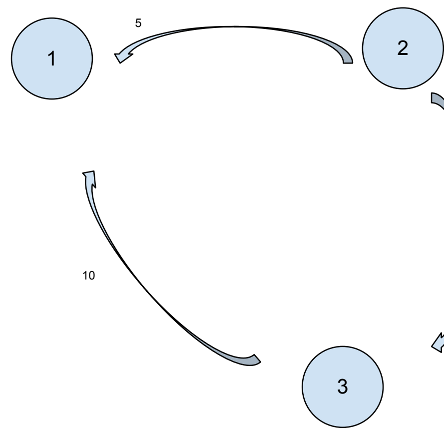


$$\begin{cases} -\pi_1 + \pi_3 + 2\pi_4 & = 0 \\ \pi_1 - 4\pi_2 & = 0 \\ 2\pi_2 - 2\pi_3 & = 0 \\ 2\pi_2 + \pi_3 - 2\pi_4 & = 0 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 & = 1 \end{cases}$$

gives:

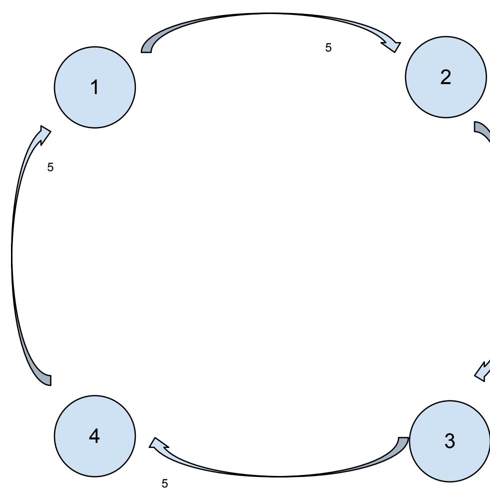
$$\pi = \left(\frac{8}{15}, \frac{2}{15}, \frac{2}{15}, \frac{3}{15}\right)$$

(f) The Chain is given:



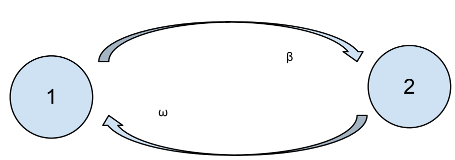
$$\pi = (1, 0, 0)$$

(g) The Chain is given:



$$\pi = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$$

(h) Only if $-\alpha = \beta \geq 0$ and $-\gamma = \omega \geq 0$



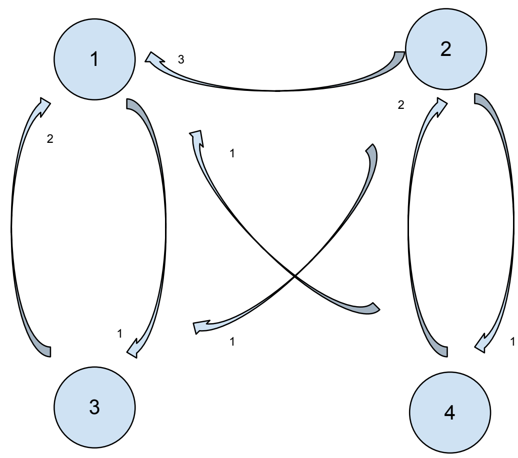
$$\begin{cases} -\beta\pi_1 + \omega\pi_2 & = 0 \\ \pi_1 + \pi_2 = 1 \end{cases}$$

Gives:

$$\pi = \left(\frac{\omega}{\beta+\omega}, \frac{\beta}{\beta+\omega}\right)$$

if $\beta + \omega = 0$ then no steady state exists.

5. Consider the following continuous Markov chain.



(a) Obtain the transition rate matrix.

$$Q = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 3 & -5 & 1 & 1 \\ 2 & 0 & -2 & 0 \\ 1 & 2 & 0 & -3 \end{pmatrix}$$

(b) Obtain the steady state probabilities for this Markov chain.

$$\begin{cases} -\pi_1 + 3\pi_2 + 2\pi_3 + \pi_4 & = 0 \\ -5\pi_2 + 2\pi_4 & = 0 \\ \pi_1 + \pi_2 - 2\pi_3 & = 0 \\ \pi_2 - 3\pi_4 & = 0 \end{cases}$$

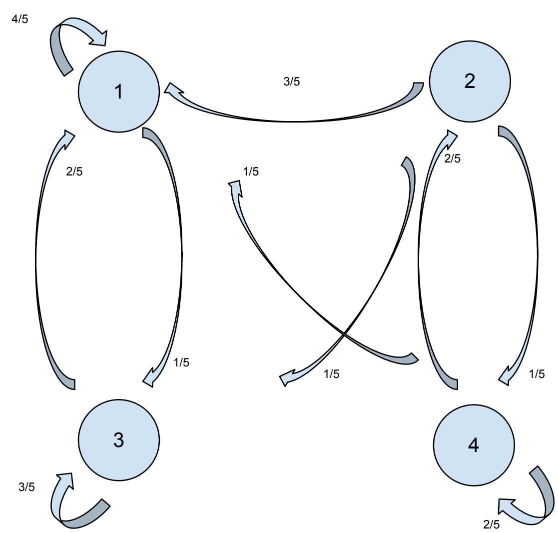
has solution:

$$\left(\frac{2}{3}, 0, \frac{1}{3}, 0\right)$$

(c) Obtain the corresponding discrete time Markov chain.
Taking $\Delta t = \frac{1}{5}$ gives:

$$P = \begin{pmatrix} \frac{4}{5} & 0 & \frac{1}{5} & 0 \\ \frac{3}{5} & 0 & \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & 0 & \frac{3}{5} & 0 \\ \frac{1}{5} & \frac{2}{5} & 0 & \frac{2}{5} \end{pmatrix}$$

(d) Draw the corresponding Markov chain.



(e) Obtain the steady state probabilities for the discretized Markov chain.

$$\begin{cases} \frac{4}{5}\pi_1 + \frac{3}{5}\pi_2 + \frac{2}{5}\pi_3 + \frac{1}{5}\pi_4 & = \pi_1 \\ \frac{2}{5}\pi_4 & = \pi_2 \\ \frac{1}{5}\pi_1 + \frac{1}{5}\pi_2 + \frac{3}{5}\pi_3 & = \pi_3 \\ \frac{1}{5}\pi_2 + \frac{2}{5}\pi_4 & = \pi_4 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 & = 1 \end{cases}$$

has solution:

$$\left(\frac{2}{3}, 0, \frac{1}{3}, 0\right)$$